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Representations of Certain Semidirect Product Groups*

JOSEPH A. WOLF

Department of Mathematics, University of California, Berkeley, California 94720

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We study a class of semidirect product groups $G = N \cdot U$ where N is a generalized Heisenberg group and U is a generalized indefinite unitary group. This class contains the Poincaré group and the parabolic subgroups of the simple Lie groups of real rank 1. The unitary representations of G and (in the unimodular cases) the Plancherel formula for G are written out. The problem of computing Mackey obstructions is completely avoided by realizing the Fock representations of N on certain U-invariant holomorphic cohomology spaces.

1. INTRODUCTION AND SUMMARY

In this paper we write out the irreducible unitary representations for a type of semidirect product group that includes the Poincaré group and the parabolic subgroups of simple Lie groups of real rank 1. If $\mathbf{F}$ is one of the four finite dimensional real division algebras, the corresponding groups in question are the $N_{p,q,F} \cdot \{\mathbf{U}(p, q; \mathbf{F}) \times \mathbf{F}^{\dagger}\}$, where

$\mathbf{F}^{p,q}$: right vector space $\mathbf{F}^{p+q}$ with hermitian form

$$h(x, y) = \sum_{1}^{p} x_i \bar{y}_i - \sum_{p+1}^{p+q} x_i \bar{y}_i ,$$

$N_{p,q,F}$: $\mathrm{Im}\, \mathbf{F} + \mathbf{F}^{p,q}$ with

$$(w_0, z_0)(w, z) = (w_0 + w + \mathrm{Im}\, h(z_0, z), z_0 + z),$$

$\mathbf{U}(p, q; \mathbf{F})$: unitary group of $\mathbf{F}^{p,q}$,

$\mathbf{F}^{\dagger}$: subgroup of the group generated by $\mathbf{F}$-scalar multiplication on $\mathbf{F}^{p,q}$.

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Here $N_{p,q,F}$ is a sort of Heisenberg group, and our main interest sits with the $G_{p,q,F} = N_{p,q,F} \cdot \mathbf{U}(p, q; \mathbf{F})$. The Poincaré group is $G_{3,1,R}$ and (for $\mathbf{F} \neq$ Cayley numbers) $G_{p,0,F} \cdot$ (multiplicative group of $\mathbf{F}$) is the parabolic subgroup of $\mathbf{U}(p+1, 1; \mathbf{F})$.

The classical procedure for enumerating the (irreducible unitary) representations of the Poincaré group goes through without serious change for all the $G_{p,q,R} = \mathbf{R}^{p,q} \cdot \mathbf{O}(p, q)$; here note $N_{p,q,R} = \mathbf{R}^{p,q}$ and $\mathbf{U}(p, q; \mathbf{R})$ is the indefinite orthogonal group $\mathbf{O}(p, q)$. In effect, every $v \in \mathbf{R}^{p,q}$ specifies a unitary character $\chi_v(z) = e^{ih(v,z)}$, χ_v has $\mathbf{O}(p, q)$-stabilizer $L_v = \{g \in \mathbf{O}(p, q): g(v) = v\}$, χ_v extends to a character $\tilde{\chi}_v$ on $\mathbf{R}^{p,q} \cdot L_v$ by $\tilde{\chi}_r(z, g) = \chi_v(z)$, one has the unitarily induced

$$\pi_{v,\gamma} = \mathrm{Ind}_{R^{p,q} \cdot L_v \uparrow G_{p,q,R}}(\tilde{\chi}_v \otimes \gamma),$$

where γ represents L_v, and every irreducible unitary representation of $G_{p,q,R}$ is equivalent to a $\pi_{v,\gamma}$. Of course v influences $\pi_{v,\gamma}$ only to the extent of its $\mathbf{O}(p, q)$-orbit, so there are just four cases: (i) $v = 0$ and $L_v = \mathbf{O}(p, q)$; (ii) $h(v, v) > 0$ and $L_v \cong \mathbf{O}(p-1, q)$; (iii) $h(v, v) < 0$ and $L_v \cong \mathbf{O}(p, q-1)$; and (iv) $v \neq 0$ with $h(v, v) = 0$, where it turns out that $L_v \cong G_{p-1,q-1,R}$. Thus one has the unitary dual $\hat{G}_{p,q,R}$ described, in several steps, in terms of the $\mathbf{O}(r, s)^\wedge$ for $0 \leqslant r \leqslant p$ and $0 \leqslant s \leqslant q$. This is known, at least in the cases $q = 0$ and $q = 1$.

This recursion procedure breaks down for the $\mathbf{F}^{p,q} \cdot \mathbf{U}(p, q; \mathbf{F})$ with $\mathbf{F}$ complex or quaternionic. In the isotropic case (iv), where $v \neq 0$ with $h(v, v) = 0$, the stabilizer L_v turns out to be $\cong G_{p-1,q-1,F}$. Working with the central extensions $G_{p,q,F}$ instead of the $\mathbf{F}^{p,q} \cdot \mathbf{U}(p, q; \mathbf{F})$ we again put ourselves in a recursive situation for the representations that arise from unitary characters of $N_{p,q,F}$. The rest of $\hat{N}_{p,q,F}$ consists of certain infinite dimensional classes $[\eta_\lambda]$, each characterized by its central character $e^{i\lambda}$,

$$\eta_\lambda(w, z) = e^{i\lambda(w)}\, \eta_\lambda(0, z),$$

where λ: Im $\mathbf{F} \to \mathbf{R}$ nonzero, linear/$\mathbf{R}$. In particular $[\eta_\lambda]$ is $\mathbf{U}(p, q; \mathbf{F})$-stable. We explicitly extend it to $G_{p,q,F}$ by using a method and theory of Satake, or results of Carmona, to realize some $\eta^{0,s} \in [\eta_\lambda]$ as the representation of $N_{p,q,F}$ on a certain square integrable cohomology group $\mathbf{H}_2^{0,s}(\mathscr{L}_\lambda)$. Here $\mathscr{L}_\lambda \to N_{p,q,F}/\mathrm{Im}\,\mathbf{F} \approx \mathbf{C}^{ap,aq}$ is the $N_{p,q,F}$-homogeneous bundle associated to $e^{i\lambda}$. This setup is $\mathbf{U}(p, q; \mathbf{F})$-stable, so $\eta^{0,s}$ extends to a representation $\tilde{\eta}_\lambda$ of $G_{p,q,F}$ on $\mathbf{H}_2^{0,s}(\mathscr{L}_\lambda)$. The resulting classes $[\tilde{\eta}_\lambda \otimes \gamma]$, $[\gamma] \in \mathbf{U}(p, q; \mathbf{F})^\wedge$, complete the description of $\hat{G}_{p,q,F}$, and they are the only classes involved when we write out the Plancherel formula for $G_{p,q,F}$.

Complete descriptions and parameterizations of the $\hat{G}_{p,q,F}$, $\mathbf{F}$ real, complex, or quaternionic, are found in Section 5. The corresponding Plancherel formulas for the $\mathbf{F}^{p,q} \cdot \mathbf{U}(p, q; \mathbf{F})$ and the $G_{p,q,F}$ are in Section 6. These results are extended in Section 7 to the $G_{p,q,F} \cdot \mathbf{F}^{\dagger}$ where $\mathbf{F}^{\dagger}$ is a subgroup of the multiplicative group of $\mathbf{F}$, except that here we do not write down any nonunimodular Plancherel formulas. The Cayley algebra case of $\mathbf{F}$, which has its own peculiarities, is found in Section 8.

Our case $q = 0$ is studied by F. W. Keene in another context, and certainly that gave me some helpful insight into the groups studied here.

There are several places where we extend representations explicitly rather than calculate the Mackey obstructions [12, 17, 18] to see whether the extensions exist. (In fact those obstructions are not very easy to compute.) Our recursive procedure combines with the explicit extensions to form a rather pretty and relatively nontechnical picture of the representation theory for our groups $G_{p,q,F}$ and $G_{p,q,F} \cdot \mathbf{F}^{\dagger}$.

2. The Groups $G_{p,q,F}$: F Real, Complex, or Quaternionic

Let $\mathbf{F}$ be a real division algebra $\mathbf{R}$ (real numbers), $\mathbf{C}$ (complex numbers) or $\mathbf{Q}$ (quaternions). We view the space $\mathbf{F}^n$ of n-tuples from $\mathbf{F}$ as a right vector space, so linear transformations act on the left. If p and q are nonnegative integers with $p + q = n$, then we have the hermitian vector space

$$\mathbf{F}^{p,q} : \mathbf{F}^n \text{ with the hermitian form } h(x, y) = \sum_{1}^{p} x^l \bar{y}^l - \sum_{p+1}^{p+q} x^l \bar{y}^l. \tag{2.1}$$

The $\mathbf{F}$-linear transformations of $\mathbf{F}^n$ that preserve h form the group

$$\mathbf{U}(p, q; \mathbf{F}) : \text{unitary group of } \mathbf{F}^{p,q}. \tag{2.2}$$

$\mathbf{U}(p, q; \mathbf{R})$ is the indefinite orthogonal group $\mathbf{O}(p, q)$, $\mathbf{U}(p, q; \mathbf{C})$ is the indefinite unitary group $\mathbf{U}(p, q)$, and $\mathbf{U}(p, q; \mathbf{Q})$ is the indefinite symplectic group $\mathbf{Sp}(p, q)$. In each case, the group is compact just when $pq = 0$, i.e. when h is positive or negative definite. The semidirect product groups

$$\bar{G}_{p,q,F} = \mathbf{F}^{p,q} \cdot \mathbf{U}(p, q; \mathbf{F}) \tag{2.3}$$

thus are generalized motion groups. Note that $\bar{G}_{1,3,R}$ is the Poincaré (inhomogeneous Lorentz) group.

As hinted in the Introduction, our initial aim was to work out the representation theory for the groups $\bar{G}_{p,q,F}$, but technical considerations forced us to work with central extensions in which the additive group of $\mathbf{F}^{p,q}$ is replaced by a 2-step nilpotent group of Heisenberg type.

Let Im $\mathbf{F}$ denote the imaginary **component** of $\mathbf{F}$, so $\mathbf{F} = \mathbf{R} + \operatorname{Im} \mathbf{F}$ as real vector space. Thus $\operatorname{Im} \mathbf{R} = 0$, $\operatorname{Im} \mathbf{C} = i\mathbf{R}$, and $\operatorname{Im} \mathbf{Q} = i\mathbf{R} + j\mathbf{R} + k\mathbf{R}$ in the usual notation. Our generalized Heisenberg groups are the

$$N_{p,q,F} = \operatorname{Im} \mathbf{F} + \mathbf{F}^{p,q} \tag{2.4a}$$

with group composition given by

$$(w_0, z_0)(w, z) = (w_0 + w + \operatorname{Im} h(z_0, z), z_0 + z). \tag{2.4b}$$

Here $w_0, w \in \operatorname{Im} \mathbf{F}$; $z_0, z \in \mathbf{F}^{p,q}$; and $\operatorname{Im} h(z_0, z)$ is the $\operatorname{Im} \mathbf{F}$-component $\frac{1}{2}\{h(z_0, z) - \overline{h(z_0, z)}\}$ of $h(z_0, z)$. Note that $N_{p,q,F}$ has center Im $\mathbf{F}$ (unless $\mathbf{F} = \mathbf{R}$) and is the simply connected group with Lie algebra

$$\mathfrak{n}_{p,q,F} = \operatorname{Im} \mathbf{F} + \mathbf{F}^{p,q} \quad \text{with} \quad [(\eta_0, \xi_0), (\eta, \xi)] = (2 \operatorname{Im} h(\xi_0, \xi), 0). \tag{2.5}$$

Of course $N_{p,q,R} = \mathbf{R}^{p,q}$, but the extension is genuine for $\mathbf{C}$ and $\mathbf{Q}$. $N_{n,0,C}$ is the ordinary Heisenberg group of dimension $2n + 1$. From (2.1) and (2.5) we have

LEMMA 2.6. *Define $f: \mathfrak{n}_{p,q,F} \to \mathfrak{n}_{p+q,0,F}$ by $f(\eta, \xi) = (\eta, \xi')$ where $\xi = (\xi_1, ..., \xi_{p+q}) \in \mathbf{F}^{p,q}$ and $\xi' = (\xi_1, ..., \xi_p, \overline{\xi_{q+1}}, ..., \overline{\xi_{p+q}})$. Then f is a real Lie algebra isomorphism, and so f induces an isomorphism of $N_{p,q,F}$ onto $N_{p+q,0,F}$.*

$\mathbf{U}(p, q; \mathbf{F})$ acts by automorphisms on $N_{p,q,F}$ by $g(w, z) = (w, g(z))$. Thus we have the semidirect product group

$$G_{p,q,F} = N_{p,q,F} \cdot \mathbf{U}(p, q; \mathbf{F}). \tag{2.7a}$$

The product in $G_{p,q,F}$ is given by

$$(w_0, z_0; g_0)(w, z; g) = (w_0 + w + \operatorname{Im} h(z_0, g_0(z)), z_0 + g_0(z), g_0 g). \tag{2.7b}$$

Evidently $G_{p,q,F}$ has center Im $\mathbf{F}$ and is a central extension $1 \to \operatorname{Im} \mathbf{F} \to G_{p,q,F} \to \bar{G}_{p,q,F} \to 1$. The point of the various p and q in $N_{p,q,F}$ is the formation of this semidirect product.

The result of Section 3 shows that $G_{p,q,F}$ occurs naturally as a subgroup of $\mathbf{U}(p+1, q+1; \mathbf{F})$, more precisely that a certain maximal parabolic subgroup $P_{p,q,F}$ of $\mathbf{U}(p+1, q+1; \mathbf{F})$ is a semidirect product $G_{p,q,F} \cdot$ (multiplicative group of $\mathbf{F}$).

In Section 8 we will discuss analogs of the $G_{p,q,F}$ and the $P_{p,q,F}$ where $\mathbf{F}$ is replaced by the (nonassociative) Cayley–Dickson division algebra.

3. $G_{p-1,q-1,F}$ as a Subgroup of $\mathbf{U}(p, q; \mathbf{F})$

In order to apply the Mackey machinery of induced representations to our semidirect product groups $\bar{G}_{p,q,F}$ and $G_{p,q,F}$ we will need

THEOREM 3.1. *Let $v \in \mathbf{F}^{p,q}$ be a nonzero isotropic ($h(v, v) = 0$; this requires $p \geqslant 1$ and $q \geqslant 1$) vector, and consider its stabilizer $L_v = \{g \in \mathbf{U}(p, q; \mathbf{F}) : g(v) = v\}$. Then L_v is isomorphic to $G_{p-1,q-1,F}$.*

We first indicate the idea of the proof. If S is a totally isotropic ($h(S, S) = 0$) subspace of $\mathbf{F}^{p,q}$, one has the group

$$P_S = \{g \in \mathbf{U}(p, q; \mathbf{F}) : g(S) = S\}.$$

The groups P_S are the maximal parabolic subgroups of $\mathbf{U}(p, q; \mathbf{F})$, and as such one knows something about their structure. L_v is a subgroup of codimension $\dim_R \mathbf{F}$ in $P_{v\mathbf{F}}$, and this will give us the structure of L_v. However, we do not actually use any theorems about parabolic subgroups in our proof of Theorem 3.1.

With the above considerations in mind, and in order to simplify notation, we denote

$$U = \mathbf{U}(p, q; \mathbf{F}) \text{ and } \mathfrak{u} \text{ is its Lie algebra,} \tag{3.2a}$$

$$P = \{g \in U : gv \in v\mathbf{F}\} \text{ and } \mathfrak{p} \text{ is its Lie algebra,} \tag{3.2b}$$

$$L = \{g \in U : gv = v\} \text{ and } \mathfrak{l} \text{ is its Lie algebra.} \tag{3.2c}$$

In terms of $\mathbf{F}$-linear transformations ξ of $\mathbf{F}^{p,q}$,

$$\mathfrak{u} = \{\xi : h(\xi z_0, z) + h(z_0, \xi z) = 0 \quad \text{for all} \quad z_0, z \in \mathbf{F}^{p,q}\}, \tag{3.3a}$$

$$\mathfrak{p} = \{\xi \in \mathfrak{u} : \xi v \in v\mathbf{F}\}, \tag{3.3b}$$

$$\mathfrak{l} = \{\xi \in \mathfrak{u} : \xi v = 0\}. \tag{3.3c}$$

Let $\{e_1, ..., e_{n=p+q}\}$ be the standard basis of $\mathbf{F}^n$, so $(z_1, ..., z_n) = \sum e_r z_r$. In the proof of Theorem 3.1 we may replace v by any $g(v)$, $g \in U$, thus replacing L and P by gLg^{-1} and gPg^{-1}; so we now assume

$$v = e_1 + e_{p+1}. \tag{3.4}$$

Now decompose $\mathbf{F}^{p,q}$ as orthogonal direct sum of its subspaces

$$V = e_1\mathbf{F} + e_{p+1}\mathbf{F} \quad \text{and} \quad W = e_2\mathbf{F} + \cdots + e_p\mathbf{F} + e_{p+2}\mathbf{F} + \cdots + e_n\mathbf{F}. \tag{3.5}$$

This allows us to decompose $\mathfrak{p}$:

LEMMA 3.6. *As real vector space, $\mathfrak{p}$ is the direct sum of*

$$\mathfrak{p}^r = \{\xi \in \mathfrak{u}: \xi W \subset W \textit{ and there exists } \gamma \in \mathbf{F} \textit{ with}$$
$$\xi e_1 = e_1(\operatorname{Im}\gamma) + e_{p+1}(\operatorname{Re}\gamma),\ \xi e_{p+1} = e_1(\operatorname{Re}\gamma) + e_{p+1}(\operatorname{Im}\gamma)\}, \tag{3.7a}$$

$$\mathfrak{p}_1{}^n = \{\xi \in \mathfrak{u}: \xi W \subset V \textit{ and } \xi V \subset W \textit{ with } \xi v = 0\}, \tag{3.7b}$$

and

$$\mathfrak{p}_2{}^n = \{\xi \in \mathfrak{u}: \xi(W) = 0 \textit{ and there exists } \beta \in \operatorname{Im}\mathbf{F} \textit{ with}$$
$$\xi e_1 = (e_1 + e_{p+1})\beta \textit{ and } \xi e_{p+1} = -(e_1 + e_{p+1})\beta\}. \tag{3.7c}$$

Further, γ is arbitrary in (3.7a) *and specified by $\xi v = v\gamma$.*

Proof. Let π_V and π_W denote projections to those subspaces. If $\xi \in \mathfrak{p}$ we decompose it as a sum $\xi^r + \xi_1{}^n + \xi_2{}^n$ as follows. First, $\xi = \xi' + \xi_1{}^n$, where

$$\xi'|_V = \pi_V \circ \xi \qquad \text{and} \qquad \xi'|_W = \pi_W \circ \xi$$

so that $\xi_1{}^n = \xi - \xi'$ is given by

$$\xi_1{}^n|_V = \pi_W \circ \xi \qquad \text{and} \qquad \xi_1{}^n|_W = \pi_V \circ \xi.$$

Then (3.3a) gives $\xi' \in \mathfrak{u}$, so also $\xi' \in \mathfrak{p}$, and thus $\xi_1{}^n \in \mathfrak{p}$. $\xi v = \xi' v$ so $\xi_1{}^n(v) = 0$. Now $\mathfrak{p}_1{}^n = \{\xi_1{}^n: \xi \in \mathfrak{p}\}$.

Second, $\xi' = \xi^r + \xi_2{}^n$, where $\xi^r|_W = \xi'|_W = \pi_W \circ \xi$ and $\xi^r|_v$ is given by

$$\xi^r e_1 = e_1(\operatorname{Im}\gamma) + e_{p+1}(\operatorname{Re}\gamma), \qquad \xi^r e_{p+1} = e_1(\operatorname{Re}\gamma) + e_{p+1}(\operatorname{Im}\gamma),$$

where $\gamma \in \mathbf{F}$ is the number defined by $\xi v = v\gamma$. Thus $\xi_2{}^n = \xi' - \xi^r$ annihilates W and $\xi_2{}^n|_V = \xi'|_V - \xi^r|_V$. Notice

$$\begin{aligned} h(\xi^r e_1, e_1) + h(e_1, \xi^r e_1) &= \operatorname{Im}\gamma + \overline{\operatorname{Im}\gamma} = 0, \\ h(\xi^r e_{p+1}, e_{p+1}) + h(e_{p+1}, \xi^r e_{p+1}) &= -\operatorname{Im}\gamma - \overline{\operatorname{Im}\gamma} = 0, \\ h(\xi^r e_1, e_{p+1}) + h(e_1, \xi^r e_{p+1}) &= -\operatorname{Re}\gamma + \operatorname{Re}\gamma = 0, \end{aligned}$$

so it follows from (3.3a) and $h(V, W) = 0$ that $\xi^r \in \mathfrak{u}$. Now $\xi^r \in \mathfrak{p}$, so also $\xi_2{}^n \in \mathfrak{p}$. Thus $\mathfrak{p}^r = \{\xi^r: \xi \in \mathfrak{p}\}$.

In the above, $\xi v = \xi' v = \xi^r v$, so $\xi_2{}^n(v) = 0$. Let $\eta = \xi_2{}^n|_V$. Since $\eta(v) = 0$, η has matrix

$$\begin{bmatrix} a & -a \\ b & -b \end{bmatrix}$$

in the basis $\{e_1, e_{p+1}\}$, and we compute

$$\begin{aligned} 0 &= h(\eta e_1, e_1) + h(e_1, \eta e_1) = a + \bar{a}, \\ 0 &= h(\eta e_{p+1}, e_{p+1}) + h(e_{p+1}, \eta e_{p+1}) = b + \bar{b}, \\ 0 &= h(\eta e_1, e_{p+1}) + h(e_1, \eta e_{p+1}) = -b - \bar{a}, \end{aligned}$$

so $a = b \in \operatorname{Im} \mathbf{F}$. Thus $\xi_2{}^n \in \mathfrak{p}_2{}^n$. It now follows that $\mathfrak{p}_2{}^n = \{\xi_2{}^n : \xi \in \mathfrak{p}\}$. Q.E.D.

We now set about examining the spaces (3.7), using them to identify $\mathfrak{p}$ and $\mathfrak{l}$ as semidirect sums $\mathfrak{p}^n + \mathfrak{p}^r$ and $\mathfrak{p}^n + (\mathfrak{l} \cap \mathfrak{p}^r)$, where $\mathfrak{p}^n = \mathfrak{p}_2{}^n + \mathfrak{p}_1{}^n$ is a nilpotent ideal $\cong \mathfrak{n}_{p-1,q-1,F}$ and where $\mathfrak{p}^r$ and $\mathfrak{l} \cap \mathfrak{p}^r$ are reductive algebras. This is summarized in Proposition 3.16 below.

LEMMA 3.8. *$\mathfrak{p}^r$ is a reductive Lie algebra, direct sum of ideals $\mathfrak{p}^r|_V$ and $\mathfrak{p}^r|_W$. Further, $\mathfrak{p}^r|_V$ is isomorphic to the Lie algebra of the multiplicative group of* $\mathbf{F}$, *and $\mathfrak{p}^r|_W = \mathfrak{l} \cap \mathfrak{p}^r$ and is naturally isomorphic to the Lie algebra of* $\mathbf{U}(p-1, q-1; \mathbf{F})$.

(This is immediate from (3.3b), (3.3c), and (3.7a).)

LEMMA 3.9. *Let $\xi \in \mathfrak{p}_1{}^n$ and express $\xi e_1 = \sum_1^n e_l z_l$. Then*

$$z_1 = z_{p+1} = 0, \qquad \xi e_{p+1} = -\sum e_l z_l,$$

and

$$\xi e_j = -(e_1 + e_{p+1})\,\bar{z}_j \text{ for } 2 \leqslant j \leqslant p, \quad +(e_1 + e_{p+1})\,\bar{z}_j \text{ for } p+2 \leqslant j \leqslant n.$$

Proof. $\xi V \subset W$ forces $z_1 = z_{p+1} = 0$, and $\xi v = 0$ forces $\xi e_{p+1} = -\xi e_1 = -\sum e_l z_l$. Since $\xi W \subset V$ we have $\xi e_j = e_1 a_j + e_{p+1} b_j$ for $1 \neq j \neq p+1$. Let $\epsilon(j) = 1$ for $j \leqslant p$, -1 for $j > p$. Then

$$0 = h(\xi e_1, e_j) + h(e_1, \xi e_j) = \epsilon(j)\, z_j + \bar{a}_j, \qquad \text{so} \qquad a_j = -\epsilon(j)\,\bar{z}_j,$$

and

$$0 = h(\xi e_{p+1}, e_j) + h(e_{p+1}, \xi e_j) = -\epsilon(j)\, z_j - \bar{b}_j, \qquad \text{so} \qquad b_j = -\epsilon(j)\,\bar{z}_j.$$

Q.E.D.

Now an element $\xi \in \mathfrak{p}_1{}^n$ is completely specified by $\xi e_1 \in W$. Note that any $z \in W$ is of this form ξe_1.

LEMMA 3.10. *Let* $\xi_1, \xi_2 \in \mathfrak{p}_1{}^n$. *Then* $[\xi_1, \xi_2] \in \mathfrak{p}_2{}^n$ *with*

$$[\xi_1, \xi_2]\, e_1 = (e_1 + e_{p+1}) \cdot 2 \operatorname{Im} \tilde{h}(\xi_1 e_1, \xi_2 e_1), \qquad \textit{where} \qquad \tilde{h}(z, z') = \overline{h(\bar{z}, \bar{z}')}.$$

Proof. Lemma 3.9 gives block form matrix expressions

$$\xi_1 = \begin{pmatrix} 0 & -{}^t\bar{x} & 0 & {}^t\bar{y} \\ x & 0 & -x & 0 \\ 0 & -{}^t\bar{x} & 0 & {}^t\bar{y} \\ y & 0 & -y & 0 \end{pmatrix} \quad \text{and} \quad \xi_2 = \begin{pmatrix} 0 & -{}^t\bar{u} & 0 & {}^t\bar{w} \\ u & 0 & -u & 0 \\ 0 & -{}^t\bar{u} & 0 & {}^t\bar{w} \\ w & 0 & -w & 0 \end{pmatrix},$$

so

$$[\xi_1, \xi_2] = \xi_1\xi_2 - \xi_2\xi_1 = \begin{pmatrix} \beta & 0 & -\beta & 0 \\ 0 & 0 & 0 & 0 \\ \beta & 0 & -\beta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} \beta &= -{}^t\bar{x} \cdot u + {}^t\bar{y} \cdot w + {}^t\bar{u} \cdot x - {}^t\bar{w} \cdot y = -2 \operatorname{Im}({}^t\bar{x} \cdot u - {}^t\bar{y} \cdot w) \\ &= -2 \operatorname{Im} h(\overline{\xi_1 e_1}, \overline{\xi_2 e_1}) = 2 \operatorname{Im} \tilde{h}(\xi_1 e_1, \xi_2 e_1). \end{aligned}$$

Q.E.D.

We record some consequences of these block form matrix expressions.

$$\text{If } \xi \in \mathfrak{p}_1{}^n \text{ and } \eta \in \mathfrak{p}_2{}^n \text{ then } \xi\eta = 0 = \eta\xi \text{ and } \xi^3 = 0. \tag{3.11a}$$

$$\text{If } \eta_1, \eta_2 \in \mathfrak{p}_2{}^n \text{ then } \eta_1\eta_2 = 0 = \eta_2\eta_1. \tag{3.11b}$$

With Lemmas 3.9 and 3.10 that gives

$$[\mathfrak{p}_1{}^n, \mathfrak{p}_1{}^n] = \mathfrak{p}_2{}^n, \qquad [\mathfrak{p}_1{}^n, \mathfrak{p}_2{}^n] = 0 \qquad \text{and} \qquad [\mathfrak{p}_2{}^n, \mathfrak{p}_2{}^n] = 0. \tag{3.12}$$

LEMMA 3.13. $\mathfrak{p}^n = \mathfrak{p}_2{}^n + \mathfrak{p}_1{}^n$ *is a Lie algebra, and we have an isomorphism* $f : \mathfrak{p}^n \to \mathfrak{n}_{p-1,q-1,F}$ *given by*

if $\eta \in \mathfrak{p}_2{}^n$ *and* $\eta e_1 = (e_1 + e_{p+1})\beta$, *then* $f(\eta) = (\bar{\beta}, 0)$

if $\xi \in \mathfrak{p}_1{}^n$ *and* $\xi e_1 = z \in W = \mathbf{F}^{p-1,q-1}$, *then* $f(\xi) = (0, \bar{z})$.

Proof. $\mathfrak{p}^n$ is a Lie algebra by (3.12). Lemma 3.9 and the definition (3.7c) show that f is an isomorphism of real vector spaces. Let $\xi_i \in \mathfrak{p}_1{}^n$ with $\xi_i e_1 = z_i$ and $\eta_i \in \mathfrak{p}_2{}^n$ with $\eta_i e_1 = (e_1 + e_{p+1})\beta_i$. Using Lemma 3.10 and (3.11) we calculate

$$\begin{aligned} f[\eta_1 + \xi_1, \eta_2 + \xi_2] &= f[\xi_1, \xi_2] = (2 \operatorname{Im} h(\bar{z}_1, \bar{z}_2), 0) \\ &= [(\bar{\beta}_1, \bar{z}_1), (\bar{\beta}_2, \bar{z}_2)] = [f(\eta_1 + \xi_1), f(\eta_2 + \xi_2)]. \end{aligned}$$

Q.E.D.

With $\mathfrak{p}^n$ identified, we turn to the action of $\mathfrak{p}^r$ on its constituents $\mathfrak{p}_2{}^n$ and $\mathfrak{p}_1{}^n$.

LEMMA 3.14. *Let* $\zeta \in \mathfrak{p}^r$, *say with* $\zeta v = v\gamma$. *If* $\xi \in \mathfrak{p}_1{}^n$, *then* $[\zeta, \xi] \in \mathfrak{p}_1{}^n$ *with* $[\zeta, \xi]e_1 = \zeta(\xi e_1) + (\xi e_1)\bar{\gamma}$.

Proof. ζ preserves V and W, and ξ interchanges them, so $[\zeta, \xi]$ interchanges them. Also $[\zeta, \xi]v = \zeta\xi v - \xi\zeta v = -\xi\zeta v = -\xi v\gamma = 0$ since $\xi v = 0$. Now $[\zeta, \xi] \in \mathfrak{p}_1{}^n$, and we use $\xi v = 0$ to calculate

$$[\zeta, \xi]\, e_1 = \zeta(\xi e_1) - \xi(\zeta e_1) = \zeta(\xi e_1) - \xi(e_1 \operatorname{Im}\gamma + e_{p+1}\operatorname{Re}\gamma)$$
$$= \zeta(\xi e_1) + \xi(e_1(\operatorname{Re}\gamma - \operatorname{Im}\gamma)) = \zeta(\xi e_1) + \xi e_1\bar{\gamma}. \qquad \text{Q.E.D.}$$

LEMMA 3.15. *Let* $\zeta \in \mathfrak{p}^r$, *say with* $\zeta v = v\gamma$. *If* $\eta \in \mathfrak{p}_2{}^n$, *then* $[\zeta, \eta] \in \mathfrak{p}_2{}^n$ *with* $[\zeta, \eta]e_1 = \gamma(\eta e_1) + (\eta e_1)\bar{\gamma}$.

Proof. ζ and η each preserve both V and W, so $[\zeta, \eta]$ preserves V and W. η annihilates $v\mathbf{F}$ and W, each of which is ζ-stable, so $[\zeta, \eta]$ annihilates $v\mathbf{F}$ and W. Now $[\zeta, \eta] \in \mathfrak{p}_2{}^n$. We compute

$$[\zeta, \eta]\, e_1 = \zeta(\eta e_1) - \eta(\zeta e_1) = \zeta(\eta e_1) - \eta(e_1 \operatorname{Im}\gamma + e_{p+1}\operatorname{Re}\gamma)$$
$$= \zeta(\eta e_1) - \eta(e_1 \operatorname{Im}\gamma - e_1 \operatorname{Re}\gamma) = \zeta(\eta e_1) + \eta(e_1)\bar{\gamma}. \qquad \text{Q.E.D.}$$

We now combine Lemmas 3.6, 3.8, 3.13, 3.14, and 3.15 with (3.11) to get the structure of the Lie algebras $\mathfrak{p}$ and $\mathfrak{l}$.

PROPOSITION 3.16. $\mathfrak{p} = \mathfrak{p}^n + \mathfrak{p}^r$ *semidirect sum where* $\mathfrak{p}^n$ *is an ideal consisting of nilpotent linear transformations and is the maximal such ideal, and where* $\mathfrak{p}^r$ *is a maximal reductive* (*completely reducible*) *subalgebra of* $\mathfrak{p}$. *Similarly* $\mathfrak{l} = \mathfrak{p}^n + (\mathfrak{l} \cap \mathfrak{p}^r)$ *semidirect sum.*

$\mathfrak{p}^n = \mathfrak{p}_2{}^n + \mathfrak{p}_1{}^n \cong \operatorname{Im}\mathbf{F} + (W = \mathbf{F}^{p-1,q-1}) = \mathfrak{n}_{p-1,q-1,F}\,.$

$\mathfrak{p}^r = \mathfrak{p}^r|_V \oplus \mathfrak{p}^r|_W$ *direct sum of ideals, where* $\mathfrak{p}^r|_V \cong \mathbf{F}$ *as vector space under* $\zeta \mapsto \gamma$ *where* $\zeta v = v\gamma$, *and* $\mathfrak{p}^r|_W = \mathfrak{l} \cap \mathfrak{p}^r$ *is the Lie algebra of* $\mathbf{U}(p-1, q-1, \mathbf{F})$, *the unitary group of* $W = \mathbf{F}^{p-1,q-1}$.

The action of $\mathfrak{p}^r|_V$ *on* $\mathfrak{p}^n$: *If* $\mathfrak{p}^r|_V \ni \zeta \leftrightarrow \gamma \in \mathbf{F}$, $\mathfrak{p}_2{}^n \ni \eta \leftrightarrow \beta \in \operatorname{Im}\mathbf{F}$ *and* $\mathfrak{p}_1{}^n \ni \xi \leftrightarrow z \in W = \mathbf{F}^{p-1,q-1}$, *then* $\mathfrak{p}_2{}^n \ni [\zeta, \eta] \leftrightarrow \gamma\beta + \beta\bar{\gamma} \in \operatorname{Im}\mathbf{F}$ *and* $\mathfrak{p}_1{}^n \ni [\zeta, \xi] \leftrightarrow z\bar{\gamma} \in W$.

The action of $\mathfrak{p}^r|_W = \mathfrak{l} \cap \mathfrak{p}^r$ *on* $\mathfrak{p}^n$: *If* $\zeta \in \mathfrak{p}^r|_W$, $\eta \in \mathfrak{p}_2{}^n$ *and* $\mathfrak{p}_1{}^n \ni \xi \leftrightarrow z \in W$, *then* $[\zeta, \eta] = 0$ *and* $\mathfrak{p}_1{}^n \ni [\zeta, \xi] \leftrightarrow \zeta(z) \in W$.

We lift Proposition 3.16 to the group level by exhibiting maximal compact subgroups of P and L. Since P and L are linear algebraic groups, a maximal compact subgroup meets every topological component.

Write $\mathbf{U}(l; \mathbf{F})$ for $\mathbf{U}(l, 0; \mathbf{F})$. Thus $\mathbf{U}(l, \mathbf{R})$ is the ordinary orthogonal group $\mathbf{O}(l)$, $\mathbf{U}(l, \mathbf{C})$ is the ordinary unitary group $\mathbf{U}(l)$, and $\mathbf{U}(l, \mathbf{Q})$ is the ordinary symplectic group $\mathbf{Sp}(l)$. They are compact.

The maximal compact subgroups of $\mathbf{U}(l, m; \mathbf{F})$ are the conjugates of $\mathbf{U}(l; \mathbf{F}) \times \mathbf{U}(m; \mathbf{F})$. Here $\mathbf{U}(l; \mathbf{F})$ acts on the first l coordinates of $\mathbf{F}^{l,m}$ and $\mathbf{U}(m; \mathbf{F})$ acts on the last m coordinates.

Write $\mathbf{F}^*$ for the multiplicative group $\{\gamma \in \mathbf{F}: \gamma \neq 0\}$. In the norm $|\gamma| = (\gamma\bar{\gamma})^{1/2}$, it has maximal compact subgroup $\mathbf{F}' = \{\gamma \in \mathbf{F}: |\gamma| = 1\}$. In fact $\gamma \mapsto (|\gamma|, |\gamma|^{-1}\gamma)$ gives an isomorphism of $\mathbf{F}^*$ onto $\mathbf{R}^+ \times \mathbf{F}'$ where $\mathbf{R}^+$ is the multiplicative group of positive real numbers. Note $\mathbf{F}' \cong \mathbf{U}(1; \mathbf{F})$. More specifically, $\mathbf{R}' = \{\pm 1\}$, and $\mathbf{C}'$ and $\mathbf{Q}'$ satisfy $\mathbf{F}' = \{e^\gamma: \gamma \in \operatorname{Im} \mathbf{F}\}$.

LEMMA 3.17. *Let K_p denote the set of all $\mathbf{F}$-linear transformations $g: \mathbf{F}^{p,q} \to \mathbf{F}^{p,q}$ such that g preserves both V and W with (identify W with $\mathbf{F}^{p-1,q-1}$)*

$g|_V$: left scalar multiplication by any element of $\mathbf{F}'$,
$g|_W$: action of any element of $\mathbf{U}(p-1; \mathbf{F}) \times \mathbf{U}(q-1; \mathbf{F})$.

Then K_P is a maximal compact subgroup of P.

Let $K_L = \{g \in K_P: g|_V = 1\} \cong \mathbf{U}(p-1; \mathbf{F}) \times \mathbf{U}(q-1; \mathbf{F})$. Then K_L is a maximal compact subgroup of L.

Proof. K_P is a compact subgroup of P, and its Lie algebra $\mathfrak{k}_P$ is the Lie algebra of a maximal compact subgroup S of P. Let $g \in S$. Then g preserves $v\mathbf{F}$, hence also $v^\perp = v\mathbf{F} + W$. As W is h-nondegenerate and $\mathfrak{k}_P$-stable, now $g(W) = W$. Thus $W^\perp = V$ is also g-stable. Now $g|_V$ is left scalar multiplication by an element of $\mathbf{F}'$, e.g. by direct calculation, and $g|_W \in \mathbf{U}(p-1; \mathbf{F}) \times \mathbf{U}(q-1; \mathbf{F})$ because it normalizes the Lie algebra of that maximal compact subgroup of $\mathbf{U}(p-1, q-1; \mathbf{F})$. So $g \in K_P$. Now $S = K_P$ so the latter is a maximal compact subgroup of P.

The assertion on K_L follows. Q.E.D.

Combining Proposition 3.16 with Lemma 3.17 we get the structure of P and L:

PROPOSITION 3.18. *There are semidirect product decompositions*

$$P = P^n \cdot P^r \quad \textit{and} \quad L = P^n \cdot (L \cap P^r)$$

where $P^n = \exp(\mathfrak{p}^n)$ *is the maximal unipotent normal subgroup, where* P^r *and* $L \cap P^r$ *have respective Lie algebras* $\mathfrak{p}^r$ *and* $\mathfrak{l} \cap \mathfrak{p}^r$, *and where* P^r *and* $L \cap P^r$ *are respective maximal reductive subgroups.*

$P^n \cong N_{p-1,q-1,F}$, *group of Heisenberg type.*

$P^r \cong \mathbf{F}^* \times \mathbf{U}(p-1, q-1; \mathbf{F})$ *with* $L \cap P^r \cong \mathbf{U}(p-1, q-1; \mathbf{F})$.

Following the isomorphism of Lemma 3.13, $\mathbf{F}^*$ *acts on* $N_{p-1,q-1,F}$ *by* $\alpha(w, z)\alpha^{-1} = (\alpha w \bar{\alpha}, \alpha z)$, *and* $\mathbf{U}(p-1, q-1; \mathbf{F})$ *acts on* $N_{p-1,q-1,F}$ *by* $g(w, z)g^{-1} = (w, \overline{g(\bar{z})})$.

Proposition 3.18 completes the proof of Theorem 3.1.

We rephrase Proposition 3.18 in the language of parabolic subgroups.

PROPOSITION 3.19. *P has Langlands decomposition MAN where the unipotent radical* $P^n = N \cong N_{p-1,q-1,F}$ *and where the reductive part* $P^r = MA \cong \mathbf{F}^* \times \mathbf{U}(p-1, q-1; \mathbf{F})$. *Identifying under those isomorphisms,*

$A = \mathbf{R}^+$ *acts on* $N_{p-1,q-1,F}$ *by* $a(w, z)a^{-1} = (a^2 w, az)$, *and* $M = M_1 \times M_2$ *where*

$M_1 = \mathbf{F}'$ *acts on* $N_{p-1,q-1,F}$ *by* $m(w, z)m^{-1} = (mw\bar{m}, mz)$,

$M_2 = \mathbf{U}(p-1, q-1; \mathbf{F})$ *acts on* $N_{p-1,q-1,F}$ *by* $g(w, z)g^{-1} = (w, \overline{g(\bar{z})})$.

$L = M_2 N$ *and so* $P = M_1 A L \cong G_{p-1,q-1,F} \cdot \mathbf{F}^*$.

4. REPRESENTATIONS OF $N_{p,q,F}$ AND THEIR STABILIZERS

We write down the irreducible unitary representations of the Heisenberg-type groups $N_{p,q,F}$. For every such representation η, we calculate the $\mathbf{U}(p, q; \mathbf{F})$-stabilizer L_η of the unitary representation class $[\eta]$, and we extend η to a unitary representation of the semidirect product $N_{p,q,F} \cdot L_\eta$ in a way that side-steps the Mackey obstruction.

From (2.4) one has $N_{p,q,F}$ as a central extension $1 \to \operatorname{Im} \mathbf{F} \to N_{p,q,F} \to \mathbf{F}^{p,q} \to 1$ where $\operatorname{Im} \mathbf{F}$ is the derived group. It follows that an irreducible unitary representation of $N_{p,q,F}$ is finite dimensional if and only if it annihilates $\operatorname{Im} \mathbf{F}$, and in that case it is a unitary character

$$e^{if}: (w, z) \mapsto e^{if(z)},$$

where $f: \mathbf{F}^{p,q} \to \mathbf{R}$ is $\mathbf{R}$-linear. The real-linear functionals on $\mathbf{F}^{p,q}$ are just the

$$f_v: \mathbf{F}^{p,q} \to \mathbf{R} \qquad \text{by} \qquad f_v(z) = \operatorname{Re} h(z, v). \tag{4.1a}$$

Now the finite dimensional irreducible unitary representations of $N_{p,q,F}$ are just the characters

$$\chi_v: (w, z) \to e^{i \operatorname{Re} h(z,v)}, \tag{4.1b}$$

where $v \in \mathbf{F}^{p,q}$.

PROPOSITION 4.2. *Fix* $v \in \mathbf{F}^{p,q}$. *If* $g \in \mathbf{U}(p, q; \mathbf{F})$, *then* $\chi_v \cdot g$ *is equivalent to* $\chi_v \Leftrightarrow \chi_v \cdot g = \chi_v \Leftrightarrow g(v) = v$. *Thus the* $\mathbf{U}(p, q; \mathbf{F})$-*stabilizer of* χ_v *is*

$$L_v = \{g \in \mathbf{U}(p, q; \mathbf{F}): g(v) = v\},$$

and there are four cases:

(i) $h(v, v) > 0$ *and* $L_v \cong \mathbf{U}(p-1, q; \mathbf{F})$;

(ii) $h(v, v) < 0$ *and* $L_v \cong \mathbf{U}(p, q-1; \mathbf{F})$;

(iii) $v = 0$ *and* $L_v = \mathbf{U}(p, q; \mathbf{F})$;

(iv) $v \neq 0$ *but* $h(v, v) = 0$, *and* $L_v \cong G_{p-1,q-1,F}$.

In each case, χ_v *extends to a unitary character on the semidirect product group* $N_{p,q,F} \cdot L_v$ *by*

$$\tilde{\chi}_v(w, z, g) = \chi_v(w, z) = e^{i \operatorname{Re} h(z,v)}. \tag{4.3}$$

Proof. Equivalence is equality for characters, and $\chi_v \cdot g = \chi_v$ just when $f_v = f_{g(v)}$, which is when $g(v) = v$.

In cases (i), (ii), and (iii), L_v is the unitary group of $v^\perp$, as specified. In case (iv), $L_v \cong G_{p-1,q-1,F}$ by Theorem 3.1.

Using (2.7b) and (4.1b), the function $\tilde{\chi}_v$ on $N_{p,q,F} \cdot L_v$ defined by (4.3) satisfies $\tilde{\chi}_v\{(w_0, z_0, g_0)(w, z, g)\} = \exp\{i \operatorname{Re} h(z_0 + g_0(z), v)\} = \exp\{i \operatorname{Re} h(z_0, v)\} \cdot \exp\{i \operatorname{Re} h(g_0(z), v)\}$. Since $g_0 \in L_v$, $h(g_0(z), v) = h(z, g_0^{-1}(v)) = h(z, v)$. Thus

$$\tilde{\chi}_v\{(w_0, z_0, g_0)(w, z, g)\} = \tilde{\chi}_v(w_0, z_0, g_0) \cdot \tilde{\chi}_v(w, z, g). \qquad \text{Q.E.D.}$$

If η is an infinite dimensional irreducible unitary representation of $N_{p,q,F}$, then $\eta|_{\operatorname{Im} \mathbf{F}}$ is of the form $\infty e^{i\lambda}$, where $e^{i\lambda}$ is a nontrivial unitary character. In other words, $\eta(w, z) = e^{i\lambda(w)} \cdot \eta(0, z)$, where $\lambda: \operatorname{Im} \mathbf{F} \to \mathbf{R}$ is nonzero and $\mathbf{R}$-linear. Of course this requires $\operatorname{Im} \mathbf{F} \neq 0$, i.e. $\mathbf{F} = \mathbf{C}$ or $\mathbf{Q}$, and then $\operatorname{Im} \mathbf{F}$ is the center of $N_{p,q,F}$ and $e^{i\lambda}$ is called the "central character" of η.

A trivial variation on the classical Heisenberg commutation relations:

LEMMA 4.4. *The equivalence classes of infinite dimensional irreducible unitary representations of* $N_{p,q,F}$ *are in bijective correspondence* $[\eta_\lambda] \leftrightarrow \lambda$ *with the nonzero* **R**-*linear functionals* λ: Im $\mathbf{F} \to \mathbf{R}$ *by*

$$\eta_\lambda(w, z) = e^{i\lambda(w)} \cdot \eta_\lambda(0, z),$$

i.e. η_λ *has central character* $e^{i\lambda}$. *The* $\mathbf{U}(p, q; \mathbf{F})$-*stabilizer of* $[\eta_\lambda]$ *is all of* $\mathbf{U}(p, q; \mathbf{F})$: *if* $g \in \mathbf{U}(p, q; \mathbf{F})$ *then* $(w, z) \mapsto \eta_\lambda(g^{-1}(w, z)g)$ *is equivalent to* η_λ.

Proof. If $\mathbf{F} = \mathbf{R}$ the Lemma is vacuously true.

Let $\mathbf{F} = \mathbf{C}$. Lemma 2.6 says $N_{p,q,C} \cong N_{n,0,C}$, $n = p + q$, which is the usual Heisenberg group of dimension $2n + 1$. The bijection $[\eta_\lambda] \leftrightarrow \lambda$ is standard for the Heisenberg group $N_{n,0,C}$ and follows for $N_{p,q,C}$. For the stabilizer: $\mathbf{U}(p, q; \mathbf{C})$ acts trivially on the center of $N_{p,q,C}$.

Let $\mathbf{F} = \mathbf{Q}$. If λ: Im $\mathbf{Q} \to \mathbf{R}$ is nonzero and **R**-linear, let $Z_\lambda = \{w \in \text{Im}\, \mathbf{Q}: \lambda(w) = 0\}$, and then $N_{p,q,Q}/Z_\lambda \cong N_{2p,2q,C}$. The case $\mathbf{F} = \mathbf{C}$ gives us an irreducible unitary representation $\bar{\eta}_\lambda$ of $N_{p,q,Q}/Z_\lambda$ whose lift η_λ to $N_{p,q,Q}$ has central character $e^{i\lambda}$. Thus $[\eta_\lambda] \mapsto \lambda$ is surjective. If η and η' are irreducible unitary representations of $N_{p,q,Q}$ with the same central character $e^{i\lambda}$, $\lambda \neq 0$, then they factor through $N_{p,q,Q}/Z_\lambda$ and give equivalent representations of that Heisenberg group, so $[\eta] = [\eta']$. Now $[\eta_\lambda] \mapsto \lambda$ is injective. For the stabilizer: $\mathbf{U}(p, q, \mathbf{Q})$ acts trivially on the center of $N_{p,q,Q}$. Q.E.D.

We now extend the infinite dimensional unitary representation classes $[\eta_\lambda]$ from $N_{p,q,F}$ to the semidirect product groups

$$N_{p,q,F} \cdot \mathbf{U}(p, q; \mathbf{F}) = G_{p,q,F}.$$

In the language of polarizations (see [2, 3]) this is done by associating $[\eta_\lambda]$ to a $\mathbf{U}(p, q; \mathbf{F})$-invariant complex polarization. However we carry out the extension in an explicit manner and thus side-step the problem of computing the Mackey obstruction [12, 17, 18].

Suppose $\mathbf{F} = \mathbf{C}$. The **R**-linear functionals λ on Im $\mathbf{C} = i\mathbf{R}$ are the

$$\lambda: \text{Im}\, \mathbf{C} \to \mathbf{R} \quad \text{by} \quad \lambda(w) = -ilw, \quad l \in \mathbf{R}; \tag{4.5a}$$

so $e^{i\lambda(ir)} = e^{ilr}$. Fixing λ, thus fixing $l \in \mathbf{R}$, we have the complex line bundle

$$\mathscr{L}_\lambda \to N_{p,q,C}/\text{Im}\, \mathbf{C} = \mathbf{C}^{p,q} \tag{4.5b}$$

associated to $e^{i\lambda}$. The C^∞ sections of $\mathscr{L}_\lambda$ are the C^∞ functions

$$\tilde{U}: N_{p,q,C} \to \mathbf{C} \tag{4.6a}$$

such that $\tilde{U}(w, z) = e^{-i\lambda(w)} \tilde{U}(0, z)$, and they are in obvious bijective correspondence with the C^∞ functions

$$U: \mathbf{C}^{p,q} \to \mathbf{C} \qquad \text{by} \qquad U(z) = \tilde{U}(0, z). \tag{4.6b}$$

$N_{p,q,C}$ acts on these sections and functions by the rule

$$\begin{aligned}[L_\lambda(w_0, z_0)\tilde{U}](w, z) &= \tilde{U}((w_0, z_0)^{-1}(w, z)) \\ &= \tilde{U}(w - w_0 - \operatorname{Im} h(z_0, z), z - z_0),\end{aligned} \tag{4.7a}$$

and thus

$$[L_\lambda(w_0, z_0)U](z) = e^{i\lambda(w_0 + \operatorname{Im} h(z_0, z))}\, U(z - z_0). \tag{4.7b}$$

The complex line bundle $\mathscr{L}_\lambda$ has the structure of holomorphic line bundle, which we now describe. Define a norm function on $\mathbf{C}^{p,q}$ and a pointwise norm on its functions by

$$\nu(z) = e^{-(l/2)h(z,z)} \qquad \text{and} \qquad \| F \|_{\lambda,z} = \nu(z)\, | F(z) |. \tag{4.8}$$

We transport the action L_λ of $N_{p,q,C}$ on C^∞ functions by the correspondence $U = \nu F$:

$$L_\lambda(w_0, z_0) \cdot (\text{multiply by } \nu) = (\text{multiply by } \nu) \cdot T_\lambda(w_0, z_0). \tag{4.9a}$$

In other words,

$$[T_\lambda(w_0, z_0)F](z) = [\nu(z - z_0)/\nu(z)] \cdot e^{i\lambda(w_0 + \operatorname{Im} h(z_0, z))} F(z - z_0). \tag{4.9b}$$

This transported action has the pleasant properties

$$\| T_\lambda(w_0, z_0)F \|_{\lambda,z} = \| F \|_{\lambda,z-z_0} \tag{4.10a}$$

and

$$T_\lambda(w_0, z_0)F \text{ is holomorphic} \Leftrightarrow F \text{ is holomorphic}. \tag{4.10b}$$

Now the holomorphic structure on $\mathscr{L}_\lambda$ is the one whose holomorphic sections are the

$$\tilde{U}: N_{p,q,C} \to \mathbf{C} \qquad \text{by} \qquad \tilde{U}(w, z) = e^{-i\lambda(w)}\, \nu(z) F(z),$$

where F is a holomorphic function on $\mathbf{C}^{p,q}$, with fibre norm

$$| \tilde{U}(w, z) | = \nu(z)\, | F(z) | = \| F \|_{\lambda,z},$$

under the action T_λ (4.9b) of $N_{p,q,C}$.

Evidently T_λ defines a unitary representation of $N_{p,q,C}$ on the Hilbert space

$$\mathbf{H}_2(\mathscr{L}_\lambda) = \left\{F\colon \mathbf{C}^{p,q} \to \mathbf{C} \text{ holomorphic: } \int_{\mathbf{C}^{p,q}} \|F\|^2_{\lambda,z}\, dz\, d\bar{z} < \infty\right\},$$

but unfortunately $\mathbf{H}_2(\mathscr{L}_\lambda) = 0$ unless lh is positive definite. Thus we have to look at the other square integrable cohomology groups of $\mathscr{L}_\lambda$.

Fix integers r, s between 0 and $n = p + q$, and denote

$$A^{r,s}(\mathscr{L}_\lambda);\ C^\infty(r, s)\text{-forms on } \mathbf{C}^{p,q} \text{ with values in } \mathscr{L}_\lambda. \tag{4.11a}$$

The hermitian metrics

$$\left\langle \sum a_{IJ}\, dz^I \wedge d\bar{z}^J, \sum b_{UV}\, dz^U \wedge d\bar{z}^V \right\rangle = \nu^2 \sum a_{IJ}\overline{b_{IJ}} \tag{4.11b}$$

($I = (i_1, \ldots, i_r)$ and $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_r}$; $J = (j_1, \ldots, j_s)$ and $d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s}$) specify Hodge–Kodaira operators

$$A^{r,s}(\mathscr{L}_\lambda) \xrightarrow{\#} A^{n-r,n-s}(\mathscr{L}_\lambda{}^* = \mathscr{L}_{-\lambda}) \xrightarrow{\tilde{\#}} A^{r,s}(\mathscr{L}_\lambda). \tag{4.11c}$$

If $\alpha, \beta \in A^{r,s}(\mathscr{L}_\lambda)$ then $\alpha \wedge \#\beta$ is an ordinary (n, n)-form on $\mathbf{C}^{p,q}$, so we have a pre-Hilbert space

$$A_2^{r,s}(\mathscr{L}_\lambda) = \left\{\alpha \in A^{r,s}(\mathscr{L}_\lambda)\colon \int_{\mathbf{C}^{p,q}} \alpha \wedge \#\alpha < \infty\right\}. \tag{4.12a}$$

The space of $\mathscr{L}_\lambda$-valued square integrable (r, s)-forms is

$$L_2^{r,s}(\mathscr{L}_\lambda)\colon \text{Hilbert space completion of } A_2^{r,s}(\mathscr{L}_\lambda). \tag{4.12b}$$

The operator

$$\bar{\partial}\colon \sum a_{IJ}\, dz^I \wedge d\bar{z}^J \mapsto (-1)^r \sum (\partial a_{IJ}/\partial \bar{z}^k)\, dz^I \wedge d\bar{z}^k \wedge d\bar{z}^J$$

gives a densely defined linear operator $\bar{\partial}\colon L_2^{r,s}(\mathscr{L}_\lambda) \to L_2^{r,s+1}(\mathscr{L}_\lambda)$, whose formal adjoint is $\bar{\partial}^* = -\tilde{\#}\bar{\partial}\#$. These give a second-order elliptic operator

$$\square = (\bar{\partial} + \bar{\partial}^*)^2 = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}\colon \text{Kodaira–Hodge–Laplace operator}, \tag{4.12c}$$

which is essentially self-adjoint from the domain of compactly supported forms in $A^{r,s}(\mathscr{L}_\lambda)$; see [1]. Its self-adjoint extension $\square^*$ (adjoint) $= \bar{\square}$ (closure) has kernel

$$\mathbf{H}_2^{r,s}(\mathscr{L}_\lambda) = \{\omega \in L_2^{r,s}(\mathscr{L}_\lambda)\colon \bar{\square}\omega = 0\}, \tag{4.13}$$

whose elements are called *square integrable harmonic* (r, s)*-forms* with values in $\mathscr{L}_\lambda$. Evidently $\mathbf{H}_2^{r,s}(\mathscr{L}_\lambda)$ is a closed subspace of $L_2^{r,s}(\mathscr{L}_\lambda)$ consisting of C^∞ forms.

$N_{p,q,C}$ acts on $A^{r,s}(\mathscr{L}_\lambda)$ by

$$T_\lambda^{r,s}(w_0, z_0) \sum a_{IJ}\, dz^I \wedge d\bar{z}^J = \sum T_\lambda(w_0, z_0)\, a_{IJ}\, dz^I \wedge d\bar{z}^J.$$

From (4.10a) that action commutes with $\#$ and $\tilde{\#}$ and thus goes over to

$$T_\lambda^{r,s}\text{: unitary representation of } N_{p,q,C} \text{ on } L_2^{r,s}(\mathscr{L}_\lambda). \tag{4.14a}$$

The action also commutes with $\bar{\partial}$ by (4.10b), so it commutes with $\Box$ and preserves $\mathbf{H}_2^{r,s}(\mathscr{L}_\lambda)$. Now $T_\lambda^{r,s}$ restricts to

$$\eta_\lambda^{r,s}\text{: unitary representation of } N_{p,q,C} \text{ on } \mathbf{H}_2^{r,s}(\mathscr{L}_\lambda). \tag{4.14b}$$

Notice that $\eta_\lambda^{r,s}$ has central character $e^{i\lambda}$. According to Satake [17] and Carmona [4],

$$\mathbf{H}_2^{0,s}(\mathscr{L}_\lambda) = 0 \text{ unless } l > 0 \text{ and } s = p \text{ or } l < 0 \text{ and } s = q \tag{4.15a}$$

and

$$\text{if } l > 0 \text{ and } s = p, \text{ or if } l < 0 \text{ and } s = q, \text{ then } \eta_\lambda^{0,s} \text{ is irreducible.} \tag{4.15b}$$

Thus $[\eta_\lambda]$, λ nontrivial, is realized by $\eta_\lambda^{0,s}$ as in (4.15b).

$\mathbf{U}(p, q; \mathbf{F})$ acts on the quotient space $\mathbf{C}^{p,q} = N_{p,q,C}/\mathrm{Im}\,\mathbf{C}$ by its ordinary linear action. This action lifts naturally to $\mathscr{L}_\lambda$ for we can view $\mathscr{L}_\lambda$ as the $G_{p,q,C}$-homogeneous line bundle over

$$\mathbf{C}^{p,q} = G_{p,q,C}/(\mathrm{Im}\,\mathbf{C}) \cdot \mathbf{U}(p, q; \mathbf{C})$$

associated to $(w, 0, g) \mapsto e^{i\lambda(w)}$. Evidently $\nu(gz) = \nu(z)$ for $z \in \mathbf{C}^{p,q}$ and $g \in \mathbf{U}(p, q; \mathbf{C})$, so the hermitian metric on $\mathscr{L}_\lambda$ is $G_{p,q,C}$-invariant, as is the holomorphic structure described after (4.10b). Since Lebesgue measure on $\mathbf{C}^{p,q}$ is $\mathbf{U}(p, q; \mathbf{C})$-invariant, now $G_{p,q,C}$ preserves all the ingredients (4.11), (4.12) in the definition (4.13) of $\mathbf{H}_2^{r,s}(\mathscr{L}_\lambda)$, and thus acts on $\mathbf{H}_2^{r,s}(\mathscr{L}_\lambda)$ by a unitary representation whose restriction to $N_{p,q,C}$ is $\eta_\lambda^{r,s}$. In view of (4.15), this extends $[\eta_\lambda]$ from $N_{p,q,C}$ to $G_{p,q,C}$. Compare Satake [17].

Finally suppose $\mathbf{F} = \mathbf{Q}$. If λ: $\mathrm{Im}\,\mathbf{Q} \to \mathbf{R}$ is nonzero and $\mathbf{R}$-linear, we set $Z_\lambda = \{w \in \mathrm{Im}\,\mathbf{Q}: \lambda(w) = 0\}$ as in the proof of Lemma 4.4. The basic units $i, j, k = ij \in \mathrm{Im}\,\mathbf{Q}$ can be chosen so that $j, k \in Z_\lambda$, and then $N_{p,q,Q}/Z_\lambda \cong N_{2p,2q,C}$ carries $\mathbf{U}(p, q; \mathbf{Q})$ over to a subgroup of $\mathbf{U}(2p, 2q, \mathbf{C})$. If $[\bar{\eta}_\lambda]$ is the unitary representation class of $N_{p,q,Q}/Z_\lambda$

that lifts to $[\eta_\lambda]$, we realize it, as above, as an $\bar{\eta}_\lambda^{0,s}$, extend $\bar{\eta}_\lambda^{0,s}$ to $G_{p,q,Q}/Z_\lambda$, and lift it to a representation of $G_{p,q,Q}$ whose $N_{p,q,Q}$-restriction is in $[\eta_\lambda]$.

In summary, we have proved

PROPOSITION 4.16. *Every infinite dimensional irreducible unitary representation class* $[\eta_\lambda]$ *of* $N_{p,q,F}$ *extends to a unitary representation class* $[\tilde{\eta}_\lambda]$ *of* $G_{p,q,F}$.

5. REPRESENTATIONS OF THE GROUPS $G_{p,q,F}$ AND $\bar{G}_{p,q,F}$

We use the results of Section 4 to apply Mackey's little-group method ([8, 9, 10, 11, 12]; see [13]) and obtain all irreducible unitary representation classes of the groups $G_{p,q,F}$ and $\bar{G}_{p,q,F}$.

First note the conditions necessary to apply the little-group method to a semidirect product $G = N \cdot U$ of locally compact groups. Write ^ for unitary dual, the set of all equivalence classes of irreducible unitary representations, with its usual Borel structure. If $[\eta] \in \hat{N}$, denote its stabilizer

$$G_\eta = N \cdot U_\eta = \{g \in G \colon n \mapsto \eta(g^{-1}ng) \text{ is equivalent to } \eta\} \tag{5.1a}$$

and consider the "extensions"

$$E(\eta) = \{[\psi] \in \hat{G}_\eta \colon \psi|_N \text{ is equivalent to a multiple of } \eta\}. \tag{5.1b}$$

If all the groups are of type I, and if there is a Borel section to the action of G on $\hat{N}$, then $\hat{G}$ consists of the classes of representations unitarily induced from $E(\eta)$, i.e.

$$\hat{G} = \{[\mathrm{Ind}_{G_\eta \uparrow G}(\psi)] \colon [\eta] \in \hat{N} \text{ and } [\psi] \in E(\eta)\}. \tag{5.1c}$$

These conditions are automatic for our groups

$$G_{p,q,F} = N_{p,q,F} \cdot \mathbf{U}(p, q; \mathbf{F})$$

because there N, G, and the G_η are algebraic and G is analytic on $\hat{N}$.

Proposition 4.2 gives us $(G_{p,q,F})_\eta$ and $E(\eta)$ when $[\eta] \in \hat{N}_{p,q,F}$ is a unitary character,

$$\eta = \chi_v \colon (w, z) \mapsto e^{i\mathrm{Re}\, h(z,v)}, \qquad v \in \mathbf{F}^{p,q}. \tag{5.2a}$$

In that case

$$(G_{p,q,F})_\eta = N_{p,q,F} \cdot L_v, \qquad L_v = \{g \in \mathbf{U}(p, q; \mathbf{F}) \colon g(v) = v\}, \tag{5.2b}$$

and η extends to a unitary character on $(G_{p,q,F})_\eta$ by the formula

$$\tilde{\eta}(w, z, g) = \eta(w, z) = e^{i\operatorname{Re}h(z,v)}. \tag{5.2c}$$

It follows that

$$E(\eta) = \{[\tilde{\eta} \otimes \gamma] : [\gamma] \in \hat{L}_v\}, \tag{5.3a}$$

where $\tilde{\eta} \otimes \gamma$ represents $(G_{p,q,F})_\eta$ on the representation space of γ by

$$(\tilde{\eta} \otimes \gamma)(w, z, g) = e^{i\operatorname{Re}h(z,v)}\, \gamma(g). \tag{5.3b}$$

The resulting induced representations

$$\pi_{v,\gamma} = \operatorname{Ind}_{N_{p,q,F}\cdot L_v \uparrow G_{p,q,F}}(\tilde{\eta} \otimes \gamma) \tag{5.4a}$$

specify a subset of the unitary dual of $G_{p,q,F}$:

$$(G_{p,q,F})_v^{\wedge} = \{[\pi_{v,\gamma}] : [\gamma] \in \hat{L}_v\}. \tag{5.4b}$$

If $v' = g(v) \in \mathbf{U}(p, q; \mathbf{F})(v)$, then the corresponding families (5.4b) coincide. On the other hand

$$\pi_{v,\gamma}\,|_{N_{p,q,F}} = \int_{\mathbf{U}(p,q;\mathbf{F})/L_v} \chi_{g(v)}\, d(gL_v),$$

so families corresponding to distinct $\mathbf{U}(p, q; \mathbf{F})$-orbits on $\mathbf{F}^{p,q}$ are disjoint. Glancing back at Proposition 4.2 we summarize as follows.

PROPOSITION 5.5. *The classes in $\hat{G}_{p,q,F}$ which arise from unitary characters on $N_{p,q,F}$ fall into four disjoint series as follows.*

1. $(G_{p,q,F})^{\wedge}_{\text{positive}}$ *is parameterized by* $\mathbf{R}^+ \times \mathbf{U}(p-1, q; \mathbf{F})^{\wedge}$ *under* $[\pi_{v,\gamma}] \leftrightarrow (r, [\gamma])$ *where* $r^2 = h(v, v) > 0$, L_v *is the* $\mathbf{U}(p-1, q; \mathbf{F})$ *acting on* $v^\perp$, *and* $[\gamma] \in \hat{L}_v$. *This series is nonvoid just when* $p > 0$.

2. $(G_{p,q,F})^{\wedge}_{\text{negative}}$ *is parameterized by* $\mathbf{R}^- \times \mathbf{U}(p, q-1; \mathbf{F})^{\wedge}$ *under* $[\pi_{v,\gamma}] \leftrightarrow (r, [\gamma])$ *where* $-r^2 = h(v, v) < 0$, L_v *is the* $\mathbf{U}(p, q-1; \mathbf{F})$ *acting on* $v^\perp$, *and* $[\gamma] \in \hat{L}_v$. *This series is nonvoid just when* $q > 0$.

3. $(G_{p,q,F})^{\wedge}_{\text{zero}}$ *is parameterized by* $\mathbf{U}(p, q; \mathbf{F})^{\wedge}$, $[\gamma]$ *corresponding to its lift* $[\pi_{0,\gamma}]$. *This series is nonvoid.*

4. $(G_{p,q,F})^{\wedge}_{\text{isotropic}}$ *is parameterized by* $\hat{G}_{p-1,q-1,F}$ *under* $[\pi_{v,\gamma}] \leftrightarrow [\gamma]$ *where* $v \neq 0$, $h(v, v) = 0$, *and* L_v *is identified with its isomorph* $G_{p-1,q-1,F}$. *This series is nonvoid just when* $p > 0$ *and* $q > 0$.

Lemma 4.4 and the discussion summarized in Proposition 4.16 give us $(G_{p,q,F})_\eta$ and $E(\eta)$ when $[\eta] \in \hat{N}_{p,q,F}$ is not a unitary character.

In that case $\eta = \eta_\lambda$, infinite dimensional representation with nontrivial central character $e^{i\lambda}(w) = e^{i\lambda(w)}$, and $(G_{p,q,F})_\eta$ is all of $G_{p,q,F}$. We have a particular extension $\tilde{\eta}_\lambda$ of η_λ to $G_{p,q,F}$, and

$$E(\eta_\lambda) = \{[\tilde{\eta}_\lambda \otimes \gamma] : [\gamma] \in \mathbf{U}(p, q; \mathbf{F})^\wedge\}, \tag{5.6a}$$

where $\tilde{\eta}_\lambda \otimes \gamma$ represents $G_{p,q,F}$ on $\mathbf{H}_{\eta_\lambda} \otimes \mathbf{H}_\gamma$ by

$$(\tilde{\eta}_\lambda \otimes \gamma)(w, z, g) = \tilde{\eta}_\lambda(w, z, g) \otimes \gamma(g). \tag{5.6b}$$

Since $G_{p,q,F}$ acts trivially on the center Im $\mathbf{F}$ of $N_{p,q,F}$, the $\tilde{\eta}_\lambda \otimes \gamma$ are mutually inequivalent. Now

PROPOSITION 5.7. *The classes in $\hat{G}_{p,q,F}$ which arise from infinite dimensional representations of $N_{p,q,F}$ form a single series*

$$(G_{p,q,F})^\wedge_\infty = \{[\tilde{\eta}_\lambda \otimes \gamma] : \ \lambda : \operatorname{Im} \mathbf{F} \to \mathbf{R} \textit{ nonzero and } [\gamma] \in \mathbf{U}(p, q; \mathbf{F})^\wedge\}.$$

This series is void for $\mathbf{F} = \mathbf{R}$ and nonvoid for $\mathbf{F} = \mathbf{C}$ and $\mathbf{F} = \mathbf{Q}$.

The Mackey little-group result (5.1) tells us that we have exhausted $\hat{G}_{p,q,F}$:

THEOREM 5.8. *The irreducible unitary representation classes of $G_{p,q,F}$ fall into five disjoint series,*

$$\hat{G}_{p,q,F} = (G_{p,q,F})^\wedge_{\text{positive}} \cup (G_{p,q,F})^\wedge_{\text{negative}} \cup (G_{p,q,F})^\wedge_{\text{zero}} \cup (G_{p,q,F})^\wedge_{\text{isotropic}} \cup (G_{p,q,F})^\wedge_\infty,$$

as described in Propositions 5.5 *and* 5.7.

View $(\bar{G}_{p,q,F})^\wedge$ as $\{[\pi] \in \hat{G}_{p,q,F} : \pi$ factors through $\bar{G}_{p,q,F}\}$.

COROLLARY 5.9. *The irreducible unitary representations of $\bar{G}_{p,q,F} = \mathbf{F}^{p,q} \cdot \mathbf{U}(p, q; \mathbf{F})$ fall into four disjoint series*

$$(\bar{G}_{p,q,F})^\wedge = (G_{p,q,F})^\wedge_{\text{positive}} \cup (G_{p,q,F})^\wedge_{\text{negative}} \cup (G_{p,q,F})^\wedge_{\text{zero}} \cup (G_{p,q,F})^\wedge_{\text{isotropic}},$$

as described in Proposition 5.5.

We now have a complete description of $\hat{G}_{p,q,F}$ modulo knowing the $\mathbf{U}(r, s; \mathbf{F})^\wedge$ for $0 \leqslant r \leqslant p$ and $0 \leqslant s \leqslant q$. This is obvious for the series $(G_{p,q,F})^\wedge_{\text{positive}}$, $(G_{p,q,F})^\wedge_{\text{negative}}$, $(G_{p,q,F})^\wedge_{\text{zero}}$ and $(G_{p,q,F})^\wedge_\infty$, so we need only explain the enumeration of $(G_{p,q,F})^\wedge_{\text{isotropic}}$. Let

$m = \min(p, q)$. Then $(G_{p-m,q-m,F})\hat{}_{\text{isotropic}}$ is empty, so we have $\hat{G}_{p-m,q-m,F}$ explicitly. This gives us $(G_{p-m+1,q-m+1,F})\hat{}_{\text{isotropic}}$, so now we have $\hat{G}_{p-m+1,q-m+1,F}$ explicitly. Continuing, we have $\hat{G}_{p-1,q-1,F}$ explicitly, it gives us $(G_{p,q,F})\hat{}_{\text{isotropic}}$, and thus we have $\hat{G}_{p,q,F}$ explicitly.

Similarly we now know $(\bar{G}_{p,q,F})\hat{}$ modulo knowing the $\mathbf{U}(r, s; \mathbf{F})\hat{}$ for $0 \leqslant r \leqslant p$ and $0 \leqslant s \leqslant q$.

One has explicit knowledge of the various series in the $\mathbf{U}(r, s; \mathbf{F})\hat{}$ which contribute to Plancherel measure, e.g. through the work of Harish–Chandra. In principle the recent work of Langlands [7] describes all of each $\mathbf{U}(r, s; \mathbf{F})\hat{}$, except that he works with Banach representations and the unitarization problem there is quite nontrivial. At any rate, one has enough for the Plancherel formulas which we write down in Section 6.

6. Plancherel Formula for the $G_{p,q,F}$ and $\bar{G}_{p,q,F}$

We combine the results of Section 5 with the Plancherel formulas for the groups $\mathbf{U}(r, s; \mathbf{F})$, $0 \leqslant r \leqslant p$ and $0 \leqslant s \leqslant q$, and write down the Plancherel formula for $G_{p,q,F}$ and $\bar{G}_{p,q,F}$.

1. Suppose $\mathbf{F} = \mathbf{R}$. This case is easily extracted from the considerations of Kleppner and Lipsman [6], who carry it out for the case $p = 1$, extending Rideau's results [16] on the Poincaré group $G_{1,3,R}$. (The just-cited authors actually work with the connected group—the identity component of our group—but passage to the larger group is a routine matter.)

The indefinite orthogonal group $\mathbf{O}(p, q) = \mathbf{U}(p, q; \mathbf{R})$ has orbits on $\mathbf{R}^{p,q} = N_{p,q,R}$ as follows.

For $r > 0$ there is the quadric $Q_r = \{v \in \mathbf{R}^{p,q}: h(v, v) = r^2\}$; (6.1a)

for $r < 0$ there is the quadric $Q_r = \{v \in \mathbf{R}^{p,q}: h(v, v) = -r^2\}$; (6.1b)

there is the origin $\{0\}$; (6.1c)

and

there is the light cone $C = \{v \in \mathbf{R}^{p,q}: v \neq 0 \text{ and } h(v, v) = 0\}$. (6.1d)

These correspond to the four series listed in Proposition 5.5. Since Lebesgue = Plancherel measure on $\mathbf{R}^{p,q}$ is concentrated in the union of the two open sets

$$\mathbf{R}_+^{p,q} = \bigcup_{r>0} Q_r \qquad \text{and} \qquad \mathbf{R}_-^{p,q} = \bigcup_{r<0} Q_r ,$$

now Plancherel measure for $G_{p,q,R} = \mathbf{R}^{p,q} \cdot \mathbf{O}(p, q)$ is concentrated in the union $(G_{p,q,R})^{\wedge}_{\text{positive}} \cup (G_{p,q,R})^{\wedge}_{\text{negative}}$ of the corresponding series. Let dQ_r denote the volume element on the quadric Q_r specified by its pseudo-riemannian metric induced from $\mathbf{R}^{p,q}$. Since $Q_{\pm|r|} = |r| Q_{-1}$, the euclidean volume element is

$$dV_{\pm} = |r|^{p+q-1} dr \wedge dQ_{\pm 1} \text{ on } \mathbf{R}^{p,q}_{\pm}.$$

Now, as in [6, pp. 511–512], $G_{p,q,R}$ has Plancherel formula

$$\int_{G_{p,q,R}} |f(x)|^2 dx = c_1 \int_0^\infty \left\{ \int_{[\gamma] \in \mathbf{O}(p-1,q)^{\wedge}} \| \pi_{v,\gamma}(f) \|_2^2 \, d[\gamma] \right\} r^{p+q-1} dr$$
$$+ c_2 \int_0^\infty \left\{ \int_{[\delta] \in \mathbf{O}(p,q-1)^{\wedge}} \| \pi_{w,\delta}(f) \|_2^2 \, d[\delta] \right\} s^{p+q-1} ds, \qquad (6.2)$$

where $h(v, v) = r^2$ and $d[\gamma]$ is Plancherel measure for $\mathbf{O}(p-1, q)$, $h(w, w) = -s^2$, and $d[\delta]$ is Plancherel measure for $\mathbf{O}(p, q-1)$, the c_i are positive constants depending on normalizations of Haar measure, and $\| \cdot \|_2$ is Hilbert–Schmidt norm.

2. Suppose $\mathbf{F} = \mathbf{C}$. As above, $\bar{G}_{p,q,C} = \mathbf{C}^{p,q} \cdot \mathbf{U}(p, q)$ has Plancherel formula

$$\int_{\bar{G}_{p,q,C}} |f(x)|^2 dx = c_1 \int_0^\infty \left\{ \int_{\mathbf{U}(p-1,q)^{\wedge}} \| \pi_{v,\gamma}(f) \|_2^2 \, d[\gamma] \right\} r^{2(p+q)-1} dr$$
$$+ c_2 \int_0^\infty \left\{ \int_{\mathbf{U}(p,q-1)^{\wedge}} \| \pi_{w,\delta}(f) \|_2^2 \, d[\delta] \right\} s^{2(p+q)-1} ds, \qquad (6.3)$$

where $h(v, v) = r^2$ and $d[\gamma]$ is Plancherel measure for $\mathbf{U}(p-1, q)$, $h(w, w) = -s^2$, and $d[\delta]$ is Plancherel measure for $\mathbf{U}(p, q-1)$, and the c_i are positive constants depending on normalization of Haar measure.

We go to $G_{p,q,C}$. The Heisenberg group $N_{p,q,C}$ has Plancherel formula

$$f(n) = c \int_{-\infty}^{\infty} \Theta_l(r_n f) \, |l|^{p+q} \, dl, \qquad f \text{ is } C^\infty \text{ and rapidly decreasing}, \qquad (6.4a)$$

where $[r_n f](n') = f(n'n)$, $c = (p+q)! \, 2^{p+q}$, and Θ_l is the distribution character of the class $[\eta_\lambda]$ of Lemma 4.4 for λ: $\operatorname{Im} \mathbf{C} \to \mathbf{R}$ by $\lambda(w) = -ilw$. Of course $\Theta_l(r_n f)$ is the orbital integral,

$$\Theta_l(r_n f) = c^{-1} \int (r_n f)^{\wedge}_1 (y) \, d\mu_l(y), \qquad (6.4b)$$

where $(r_n f)_1$ is the function $(r_n f) \cdot \exp$ on the Lie algebra $\mathfrak{n}_{p,q,C}$, $(r_n f)_1^{\wedge}$ is its Fourier transform (function on the real dual space $\mathfrak{n}_{p,q,C}^*$), the integration is over the $Ad(N_{p,q,C})^*$-orbit of $\lambda: \mathfrak{n}_{p,q,C} \to \mathbf{R}$ by $\lambda(w, z) = -ilw$, and $d\mu_l$ is symplectic measure (see [15]).

Now the Plancherel measure for $G_{p,q,C} = N_{p,q,C} \cdot \mathbf{U}(p, q)$ is concentrated on the series $(G_{p,q,C})_\infty^{\wedge}$ of representation classes described in Proposition 5.7. Applying [6, Theorem 2.3] we get the Plancherel formula for $G_{p,q,C}$,

$$\int_{G_{p,q,C}} |f(x)|^2 \, dx = c' \int_{-\infty}^{\infty} \left\{ \int_{[\gamma] \in \mathbf{U}(p,q)^{\wedge}} \|(\tilde{\eta}_\lambda \otimes \gamma)(f)\|^2 \, d[\gamma] \right\} |l|^{p+q} \, dl, \tag{6.5}$$

where $\lambda(w) = -ilw$ as above, $d[\gamma]$ is Plancherel measure for $\mathbf{U}(p, q)$, and c' is a positive constant depending on normalization of Haar measures.

3. Suppose $\mathbf{F} = \mathbf{Q}$. As in (6.2), the group $\tilde{G}_{p,q,Q} = \mathbf{Q}^{p,q} \cdot \mathbf{Sp}(p, q)$ has Plancherel formula

$$\int_{\tilde{G}_{p,q,Q}} |f(x)|^2 \, dx = c_1 \int_0^{\infty} \left\{ \int_{\mathbf{Sp}(p-1,q)^{\wedge}} \| \pi_{v,\gamma}(f)\|_2^2 \, d[\gamma] \right\} r^{4(p+q)-1} \, dr$$

$$+ c_2 \int_0^{\infty} \left\{ \int_{\mathbf{Sp}(p,q-1)^{\wedge}} \| \pi_{w,\delta}(f)\|_2^2 \, d[\delta] \right\} s^{4(p+q)-1} \, ds, \tag{6.6}$$

where $h(v, v) = r^2$, $h(w, w) = -s^2$, etc.

We go to $G_{p,q,Q}$. The Heisenberg type group $N_{p,q,Q}$ has Plancherel formula

$$f(n) = c \int_{\mathbf{R}^3} \Theta_l(r_n f) \| l \|^{2(p+q)} \, dl, \qquad f \text{ is } C^\infty \text{ and rapidly decreasing,} \tag{6.7a}$$

where $c = (2p + 2q)! \, 2^{2(p+q)}$, dl is Lebesgue measure on $\mathbf{R}^3$, and Θ_l is the distribution character of the class $[\eta_\lambda]$ of Lemma 4.4 for $\lambda: \operatorname{Im} \mathbf{Q} \to \mathbf{R}$ by $\lambda(a_1 i + a_2 j + a_3 k) = l_1 a_1 + l_2 a_2 + l_3 a_3$. There $\Theta_l(r_n f)$ is the orbital integral

$$\Theta_l(r_n f) = c^{-1} \int (r_n f)_1^{\wedge} (y) \, d\mu_l(y), \tag{6.7b}$$

where μ_l is symplectic measure on the $Ad(N_{p,q,Q})^*$-orbit of $\lambda: \mathfrak{n}_{p,q,Q} \to \mathbf{R}$ by $\lambda(a_1 i + a_2 j + a_3 k, z) = l_1 a_1 + l_2 a_2 = l_3 a_3$ (see [14]).

Now the Plancherel measure for $G_{p,q,Q} = N_{p,q,Q} \cdot \mathbf{Sp}(p, q)$ is concentrated on the series $(G_{p,q,Q})_\infty^{\wedge}$ of representation classes described in Proposition 5.7. As before [6, Theorem 2.3] gives us the Plancherel formula for $G_{p,q,Q}$:

$$\int_{G_{p,q,Q}} |f(x)|^2 \, dx = c' \int_{\mathbf{R}^3} \left\{ \int_{[\gamma] \in \mathbf{Sp}(p,q)^\wedge} \|(\tilde{\eta}_\lambda \otimes \gamma)(f)\|_2^2 \, d[\gamma] \right\} \| l \|^{2(p+q)} \, dl, \quad (6.8)$$

where $\lambda(a_1 i + a_2 j + a_3 k) = l_1 a_1 + l_2 a_2 + l_3 a_3$ as above, $d[\gamma]$ is Plancherel measure for $\mathbf{Sp}(p, q)$, and c' is a positive constant depending on normalization of Haar measures.

7. Representations of Certain Parabolic Groups

The results of Section 3 show that $G_{p,q,F}$ is contained in a certain maximal parabolic subgroup $P = P_{p,q,F}$ of $\mathbf{U}(p + 1, q + 1; \mathbf{F})$. There P has Langlands decomposition MAN with

$$M = \mathbf{F}' \times \mathbf{U}(p, q; \mathbf{F}), \qquad A = \mathbf{R}^+, \qquad \text{and} \qquad N = N_{p,q,F}, \quad (7.1a)$$

which naturally leads one to consider its nonreductive subgroups

$$\begin{aligned} N &= N_{p,q,F}, & AN &= N_{p,q,F} \cdot \mathbf{R}^+, \\ MN &= G_{p,q,F} \cdot \mathbf{F}', & \text{and} \qquad MAN &= G_{p,q,F} \cdot \mathbf{F}^*. \end{aligned} \quad (7.1b)$$

Here we use the method of Sections 5 and 6 to describe the representations of AN, MN, and $P = MAN$, and the Plancherel formula for MN. See the thesis of F. W. Keene [5] for the case $q = 0$. The Plancherel formulas for the nonunimodular groups AN and MAN are more delicate. In view of Lemma 2.6, the formula is worked out by Keene [5] for AN. Keene and I plan to write it out for MAN in a joint paper.

We first consider representations that come from unitary characters on N. Here we retain the notation of Propositions 4.2 and 5.5. Recall from Proposition 3.18 that $\mathbf{F}^* = \mathbf{F}' \times \mathbf{R}^+$ acts on $G_{p,q,F} = N_{p,q,F} \cdot \mathbf{U}(p, q; \mathbf{F})$ by

$$a(w, z, g)a^{-1} = (aw\bar{a}, az, g), \qquad 0 \neq a \in \mathbf{F}. \quad (7.2)$$

In particular $\mathbf{F}^*$ acts on the unitary characters of N by

$$\begin{aligned} [a(\chi_v)](w, z) &= [a(\chi_v)](0, z) = \chi_v(a^{-1}(0, z)a) = \chi_v(0, a^{-1}z) \\ &= \exp(i \operatorname{Re} h(v, a^{-1}z)) = \exp(i \operatorname{Re} h(\bar{a}^{-1}v, z)) = \chi_{\bar{a}^{-1}v}(0, z) \\ &= \chi_{\bar{a}^{-1}v}(w, z), \end{aligned}$$

that is

$$\text{if } a \in \mathbf{F}^* \quad \text{and} \quad v \in \mathbf{F}^{p,q}, \quad \text{then} \quad a(\chi_v) = \chi_{\bar{a}^{-1}v}\,. \tag{7.3a}$$

If $\mathbf{F}^\dagger$ is a subgroup $\mathbf{F}^*$, now χ_v has $\mathbf{U}(p, q; \mathbf{F}) \times \mathbf{F}^\dagger$-stabilizer

$$L_v{}^\dagger = \{(g, a) \in \mathbf{U}(p, q; \mathbf{F}) \times \mathbf{F}^\dagger : g(\bar{v}) = \bar{v}a^{-1}\}. \tag{7.3b}$$

As in (4.3), χ_v extends to $N_{p,q,F} \cdot L_v{}^\dagger$ by the formula

$$\tilde{\chi}_v(w, z, g, a) = \chi_v(w, z) = e^{i\operatorname{Re} h(z,v)}. \tag{7.3c}$$

Notice $L_v{}^\dagger \cong \{g \in \mathbf{U}(p, q; \mathbf{F}) : g\bar{v} \in \bar{v}\mathbf{F}^\dagger\}$ under $g \mapsto (g, a)$, where $g\bar{v} = \bar{v}a^{-1}$. Thus

$$\text{if } h(v, v) > 0, \quad \text{then} \quad L_v{}^\dagger \cong \mathbf{U}(p-1, q; \mathbf{F}) \times (\mathbf{F}^\dagger \cap \mathbf{F}'), \tag{7.4a}$$

$$\text{if } h(v, v) < 0, \quad \text{then} \quad L_v{}^\dagger \cong \mathbf{U}(p, q-1; \mathbf{F}) \times (\mathbf{F}^\dagger \cap \mathbf{F}'), \tag{7.4b}$$

$$\text{if } v = 0, \quad \text{then} \quad L_v{}^\dagger = \mathbf{U}(p, q; \mathbf{F}) \times \mathbf{F}^\dagger, \tag{7.4c}$$

$$\begin{cases} \text{if } v \neq 0 \text{ but } h(v, v) = 0, \text{ then } L_v{}^*(\mathbf{F}^\dagger = \mathbf{F}^*) \cong P_{p-1,q-1,F} \\ \text{and } L_v{}'(\mathbf{F}^\dagger = \mathbf{F}') \text{ is the } MN \text{ of its Langlands decomposition.} \end{cases} \tag{7.4d}$$

Now we have the extensions of χ_v to $N_{p,q,F} \cdot L_v{}^\dagger$, and thus the induced representations of $G_{p,q,F} \cdot \mathbf{F}^\dagger$, in particular for $\mathbf{F}^\dagger = \mathbf{R}^+$, $\mathbf{F}'$, or $\mathbf{F}^*$.

Now consider representations that do not come from unitary characters on N. Here we retain the notation of Lemma 4.4 and Propositions 4.16 and 5.7. Use (7.2) to trace the action of $\mathbf{F}^*$ on $\{[\eta_\lambda] : \lambda : \operatorname{Im} \mathbf{F} \to \mathbf{R} \text{ is nonzero and } \mathbf{R}\text{-linear}\}$ by means of the central character:

$$[a(\eta_\lambda)](w, 0) = \eta(aw\bar{a}, 0) = \eta_{a(\lambda)}(w, 0), \qquad \text{where} \qquad [a(\lambda)]w = \lambda(aw\bar{a}),$$

that is

$$\text{if } a \in \mathbf{F}^* \quad \text{and} \quad \lambda : \operatorname{Im} \mathbf{F} \to \mathbf{R}, \quad \text{then} \quad a[\eta_\lambda] = \eta_{a(\lambda)}\,, \tag{7.5a}$$

where $[a(\lambda)]w = \lambda(aw\bar{a})$.

Now $[\eta_\lambda]$ has $\mathbf{U}(p, q; \mathbf{F}) \times \mathbf{F}^*$- and $\mathbf{U}(p, q; \mathbf{F}) \times \mathbf{F}'$-stabilizers

$$L'_{[\eta_\lambda]} = \mathbf{U}(p, q; \mathbf{F}) \times \{a \in \mathbf{F}' : a \in \mathbf{R} + \mathbf{R}l, \qquad \text{where}$$

$$\lambda(w) = -\operatorname{Re}(l\bar{w}),\ l \in \operatorname{Im} \mathbf{F}\}. \tag{7.5b}$$

$[\eta_\lambda]$ extends to $N_{p,q,F} \cdot L'_{[\eta_\lambda]}$ as in the discussion preceding Proposition 4.16. In both cases ($\mathbf{F} = \mathbf{C}$ and $\mathbf{F} = \mathbf{Q}$; $\mathbf{R}$ does not occur here), $L'_{[\eta_\lambda]} \cong \mathbf{U}(p, q; \mathbf{F}) \times \mathbf{C}'$. This tells us the extensions of $[\eta_\lambda]$ to $N_{p,q,F} \cdot L'_{[\eta_\lambda]}$, and thus the induced representations of $G_{p,q,F} \cdot \mathbf{F}^*$ and $G_{p,q,F} \cdot \mathbf{F}'$.

In summary, Mackey's little-group method gives us the unitary duals of $AN = N_{p,q,F} \cdot \mathbf{R}^+$, $MN = G_{p,q,F} \cdot \mathbf{F}'$ and $P = MAN = G_{p,q,F} \cdot \mathbf{F}^*$ as described in Propositions 7.6 and 7.8 below.

PROPOSITION 7.6. *Let $\mathbf{F}^\dagger$ denote $\mathbf{R}^+$, $\mathbf{F}'$ or $\mathbf{F}^*$. Then $(N_{p,q,F} \cdot \mathbf{F}^\dagger)^\wedge$ is the disjoint union of three series, as follows.*

(1) *The series $\hat{\mathbf{F}}^\dagger$ viewed as the representation classes that annihilate the normal subgroup $N_{p,q,F}$;*

(2) *a series of classes $[\mathrm{Ind}_{N \uparrow N\mathbf{F}^\dagger}(\chi_v)]$, $0 \neq v \in \mathbf{F}^{p,q}$, parameterized by the space of orbits $v\mathbf{F}^\dagger$ in $\mathbf{F}^{p,q}$;*

(3) *a series of classes $[\mathrm{Ind}_{NS_\lambda \uparrow N\mathbf{F}^\dagger}(\tilde{\eta}_\lambda \otimes \sigma)]$, $\lambda\colon \mathrm{Im}\,\mathbf{F} \to \mathbf{R}$ by $\lambda(w) = -\mathrm{Re}(l\bar{w})$ where $0 \neq l \in \mathrm{Im}\,\mathbf{F}$ and where*

$$S_\lambda = \{a \in \mathbf{F}^\dagger \cap \mathbf{F}'\colon al = la\},$$

series parameterized by $\{(\mathrm{Im}\,\mathbf{F} - \{0\})/\mathbf{F}^\dagger$ under the action $a\colon l \mapsto al\bar{a}\} \times \hat{S}_\lambda$.

In regard to the series (3) above, here is a case-by-case explicit parameterization. The series exists only for $F = \mathbf{C}$ and for $\mathbf{F} = \mathbf{Q}$. If $\mathbf{F}^\dagger = \mathbf{R}^+$ then $S_\lambda = \{1\}$; otherwise S_λ is the circle group $\mathbf{F}' \cap (\mathbf{R} + \mathbf{R}l)$ and $\hat{S}_\lambda$ is parameterized by $\mathbf{Z}$. The orbit structure of $\mathrm{Im}\,\mathbf{F} - \{0\}$ under the action $a\colon l \mapsto al\bar{a}$ of $\mathbf{F}^\dagger$ is

(7.7)

	$\mathbf{F}^\dagger = \mathbf{R}^+$	$\mathbf{F}^\dagger = \mathbf{F}'$	$\mathbf{F}^\dagger = \mathbf{F}^*$
$\mathbf{F} = \mathbf{C}$	orbits $i\mathbf{R}^+$ and $-i\mathbf{R}^+$, orbit space $\{1, -1\}$	one-point orbits, orbit space $\mathrm{Im}\,\mathbf{C} - \{0\}$	orbits $i\mathbf{R}^+$ and $-i\mathbf{R}^+$, orbit space $\{1, -1\}$
$\mathbf{F} = \mathbf{Q}$	orbits $l\mathbf{R}^+$ real rays, orbit space $P^2(\mathbf{R})$ real projective plane	2-sphere orbits $\lvert l \rvert = r$, $r > 0$, orbit space $\mathbf{R}^+$	one orbit $\mathrm{Im}\,\mathbf{Q} - \{0\}$, orbit space $\{\text{point}\}$

PROPOSITION 7.8. *Let $\mathbf{F}^\dagger$ denote $\mathbf{R}^+$, $\mathbf{F}'$, or $\mathbf{F}^*$. Then $(G_{p,q,F} \cdot \mathbf{F}^\dagger)^\wedge$ is the disjoint union of five series, as follows.*

(1) *The series $\{\mathbf{U}(p, q; \mathbf{F}) \times \mathbf{F}^\dagger\}^\wedge$ viewed as the representation classes that annihilate $N_{p,q,F}$;*

(2) *the series of classes*

$$[\pi_{v,\gamma,\beta}] = [\mathrm{Ind}_{NL_v{}^\dagger \uparrow G_{p,q,F}\mathbf{F}^\dagger}(\tilde{\chi}_v \otimes \gamma \otimes \beta)],$$

$[\pi_{v,\gamma}] \in (G_{p,q,F})\hat{}_{\text{positive}}$ *and* $[\beta] \in (\mathbf{F}^\dagger \cap \mathbf{F}')\hat{}$, *series parameterized by* $\mathbf{F}^\dagger = \mathbf{F}'$: $\mathbf{R}^+ \times \mathbf{U}(p-1,\ q;\ \mathbf{F})\hat{} \times \hat{\mathbf{F}}'$ *under* $[\pi_{v,\gamma,\beta}] \leftrightarrow (h(v,v)^{1/2}, [\gamma], [\beta])$, $\mathbf{F}^\dagger = \mathbf{F}^*$: $\mathbf{U}(p-1,\ q;\ \mathbf{F})\hat{} \times \hat{\mathbf{F}}'$ *under* $[\pi_{v,\gamma,\beta}] \leftrightarrow ([\gamma], [\beta])$, $\mathbf{F}^\dagger = \mathbf{R}^+$: $\mathbf{U}(p-1, q; \mathbf{F})\hat{}$ *under* $[\pi_{v,\gamma,\beta}] \leftrightarrow [\gamma]$ *where* $h(v,v) > 0$ *and* v *has* $\mathbf{U}(p, q; \mathbf{F}) \times \mathbf{F}^\dagger$ *stabilizer* $L_v{}^\dagger$ *given by* (7.4a);

(3) *the series of classes* $[\pi_{v,\gamma,\beta}]$, $[\pi_{v,\gamma}] \in (G_{p,q,F})\hat{}_{\text{negative}}$, *and* $[\beta] \in (\mathbf{F}^\dagger \cap \mathbf{F}')\hat{}$, *series parameterized by* $\mathbf{F}^\dagger = \mathbf{F}'$: $\mathbf{R}^- \times \mathbf{U}(p, q-1; \mathbf{F})\hat{} \times \hat{\mathbf{F}}'$ *under* $[\pi_{v,\gamma,\beta}] \leftrightarrow (-\mid h(v,v)\mid^{1/2}, [\gamma], [\beta])$, $\mathbf{F}^\dagger = \mathbf{F}^*$: $\mathbf{U}(p, q-1; \mathbf{F})\hat{} \times \hat{\mathbf{F}}'$ *under* $[\pi_{v,\gamma,\beta}] \leftrightarrow ([\gamma], [\beta])$, $\mathbf{F}^\dagger = \mathbf{R}^+$: $\mathbf{U}(p, q-1; \mathbf{F})\hat{}$ *under* $[\pi_{v,\gamma,\beta}] \leftrightarrow [\gamma]$ *where* $h(v,v) < 0$ *and* v *has* $\mathbf{U}(p, q; \mathbf{F}) \times \mathbf{F}^\dagger$ *stabilizer* $L_v{}^\dagger$ *given by* (7.4b);

(4) *the series of classes* $[\pi_{v,\gamma,\beta}]$, $[\pi_{v,\gamma}] \in (G_{p,q,F})\hat{}_{\text{isotropic}}$, *and* $[\beta] \in \hat{F}^\dagger$, *series parameterized by* $\hat{G}_{p-1,q-1,F} \times \mathbf{F}^\dagger$ *under* $[\pi_{v,\gamma,\beta}] \leftrightarrow ([\gamma], [\beta])$, *where* $0 \neq v \in \mathbf{F}^{p,q}$ *with* $h(v,v) = 0$ *and* v *has* $\mathbf{U}(p, q; \mathbf{F}) \times \mathbf{F}^\dagger$-*stabilizer* $L_v{}^\dagger \cong G_{p-1,q-1,F} \times \mathbf{F}^\dagger$ *as in* (7.4d);

(5) *the series of classes*

$$[\pi_{\lambda,\gamma,\sigma}] = [\mathrm{Ind}_{NL_\lambda{}^\dagger \uparrow G_{p,q,F}\mathbf{F}^\dagger}(\tilde{\eta}_\lambda \otimes \gamma \otimes \sigma)],$$

λ: $\mathrm{Im}\,\mathbf{F} \to \mathbf{R}$ *by* $\lambda(w) = -\mathrm{Re}(l\bar{w})$, *where* $0 \neq l \in \mathrm{Im}\,\mathbf{F}$,

$$S_\lambda = \{a \in F^\dagger \cap F' : al = la\},$$

and $L_\lambda{}^\dagger = \mathbf{U}(p, q; \mathbf{F}) \times S_\lambda$, *series parameterized by*

$$\{(\mathrm{Im}\,\mathbf{F} - \{0\})/\mathbf{F}^\dagger \text{ under the action } a: l \mapsto al\bar{a}\} \times \mathbf{U}(p, q; \mathbf{F})\hat{} \times \hat{S}_\lambda$$

under $[\pi_{\lambda,\gamma,\sigma}] \leftrightarrow (\{al\bar{a} : a \in F^\dagger\}, [\gamma], [\sigma])$; *see* (7.7).

The Plancherel measure of $MN = G_{p,q,F} \cdot \mathbf{F}'$ is concentrated on the series (5) of classes $[\pi_{\lambda,\gamma,\sigma}]$ in Proposition 7.8 when $\mathbf{F} \neq \mathbf{R}$, on the union of series (2) and series (3) when $\mathbf{F} = \mathbf{R}$. As in Section 6, the corresponding Plancherel formulas are as follows.

If $\mathbf{F} = \mathbf{R}$, then $\mathbf{F}' = \mathbf{R}' = \{\pm 1\}$ so $\mathbf{F}' = \{\epsilon_\pm\}$, where $\epsilon_\pm(1) = 1$ and $\epsilon_\pm(-1) = \pm 1$, and (6.2) goes over to

$$\left\{\begin{aligned} &\int_{G_{p,q,R}\cdot\mathbf{R}'} |f(x)|^2\,dx \\ &\quad = c_1 \int_0^\infty \left\{\int_{[\gamma]\in\mathbf{O}(p-1,q)\hat{}} (\|\pi_{v,\gamma,\epsilon_+}(f)\|_2^2 + \|\pi_{v,\gamma,\epsilon_-}(f)\|_2^2)\,d[\gamma]\right\} r^{p+q-1}\,dr \\ &\qquad + c_2 \int_0^\infty \left\{\int_{[\delta]\in\mathbf{O}(p,q-1)\hat{}} (\|\pi_{w,\delta,\epsilon_+}(f)\|_2^2 + \|\pi_{w,\delta,\epsilon_-}(f)\|_2^2)\,d[\delta]\right\} s^{p+q-1}\,ds. \end{aligned}\right. \tag{7.9}$$

If $\mathbf{F} = \mathbf{C}$, then $\mathbf{F}' = \mathbf{C}' = \{e^{i\theta}: \theta \text{ real}\}$ so $\hat{\mathbf{F}}'$ consists of the characters $\zeta_n: e^{i\theta} \mapsto e^{in\theta}$, $n \in \mathbf{Z}$, and (6.5) goes over to

$$\left\{\begin{aligned} &\int_{G_{p,q,C}\cdot\mathbf{C}'} |f(x)|^2\,dx \\ &\quad = c' \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{\int_{[\gamma]\in\mathbf{U}(p,q)^{\prime\prime}} \|\pi_{\lambda,\gamma,\zeta_n}(f)\|_2^2\, d[\gamma]\right\} |l|^{p+q}\, dl. \end{aligned}\right. \tag{7.10}$$

If $\mathbf{F} = \mathbf{Q}$ we still have S_λ a circle group in Proposition 7.8 for $\mathbf{F}^\dagger = \mathbf{F}'$, and so, using (7.7), the formula (6.8) goes over to

$$\left\{\begin{aligned} &\int_{G_{p,q,Q}\cdot\mathbf{Q}'} |f(x)|^2\,dx \\ &\quad = c'' \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} \left\{\int_{[\gamma]\in\mathbf{Sp}(p,q)^{\frown}} \|\pi_{\lambda_r,\gamma,\zeta_n}(f)\|_2^2\, d[\gamma]\right\} r^{2(p+q)}\, dr, \end{aligned}\right. \tag{7.11}$$

where λ_r: $\operatorname{Im}\mathbf{Q} \to \mathbf{R}$ by $\lambda_r(w) = -\operatorname{Re}(ri\bar{w})$, $r > 0$, so $S_{\lambda_r} = \mathbf{C}'$.

8. The Cayley Variation on Our Theme

Our considerations of the semidirect product groups $G_{p,q,F}$ and their enveloping parabolics $P_{p,q,F} = G_{p,q,F} \cdot \mathbf{F}^*$ go through to some extent when $\mathbf{F}$ is replaced by the Cayley–Dickson division algebra **Cay**. These new groups are of some intrinsic interest; for example $P_{1,0,\mathbf{Cay}}$ is the minimal parabolic subgroup of the group of type F_4 with maximal compact subgroup $\mathbf{Spin}(9)$, associated to the Cayley hyperbolic plane, and has been studied from this viewpoint in Keene's thesis [5].

$\mathbf{Cay}^n$ denotes the real vector space of n-tuples of Cayley numbers, and as in (2.1) we denote

$$\mathbf{Cay}^{p,q}: \mathbf{Cay}^{p+q} \text{ with the ``hermitian'' form } h(x, y) = \sum_1^p x^l \bar{y}^l - \sum_{p+1}^{p+q} x^l \bar{y}^l. \tag{8.1}$$

This defines a Heisenberg-type group

$$N_{p,q,\mathbf{Cay}} = \operatorname{Im}\mathbf{Cay} + \mathbf{Cay}^{p,q} \text{ with group composition given by} \tag{8.2a}$$

$$(w_0, z_0)(w, z) = (w_0 + w + \operatorname{Im} h(z_0, z), z_0 + z). \tag{8.2b}$$

$N_{p,q,\mathbf{Cay}}$ is a simply connected nilpotent Lie group of real dimension $8(p+q)+7$ with 7-dimensional center $\operatorname{Im}\mathbf{Cay}$. It has Lie algebra

$$\mathfrak{n}_{p,q,\mathbf{Cay}} = \operatorname{Im}\mathbf{Cay} + \mathbf{Cay}^{p,q} \quad \text{with} \quad [(\eta_0, \xi_0), (\eta, \xi)] = (2 \operatorname{Im} h(\xi_0, \xi), 0). \tag{8.2c}$$

Lemma 2.6 goes through without change to these groups.

We must be careful as to just what we mean by

$$\mathrm{U}(p, q; \mathbf{Cay})\text{: unitary group of } \mathbf{Cay}^{p,q}. \tag{8.3}$$

We want it to consist of the **Cay**-linear transformations that preserve h. By definition, a transformation

$$T\colon \mathbf{Cay}^n \to \mathbf{Cay}^m \tag{8.4a}$$

is **Cay**-linear if it is **R**-linear and satisfies

$$T(z)a = T(za) \quad \text{for} \quad z \in \mathbf{Cay}^n \quad \text{and} \quad a \in \mathbf{Cay}, \tag{8.4b}$$

i.e. if it commutes with all right "scalar" multiplications

$$r(a)\colon (z_1, \ldots, z_l) \mapsto (z_1 a, \ldots, z_l a)$$

in the sense $T \cdot r(a) = r(a) \cdot T$ for all $a \in \mathbf{Cay}$.

LEMMA 8.5. *$T\colon \mathbf{Cay}^n \to \mathbf{Cay}^m$ is* **Cay***-linear if and only if $T(z_1, \ldots, z_n) = (\sum_1^n a_{1j}z_j, \ldots, \sum_1^n a_{mj}z_j)$ for some $m \times n$ real matrix (a_{ij}).*

Proof. First consider the case $m = n$. Let **A** denote the real associative algebra of **R**-linear transformations of $\mathbf{Cay}^n$ generated by the $r(a)$, $a \in \mathbf{Cay}$. Since $x \cdot \mathbf{Cay} = \mathbf{Cay}$ for $0 \neq x \in \mathbf{Cay}$, the real vector space of $\mathbf{Cay}^n$ is direct sum of n irreducible **A**-modules. Now **A** is a simple associative algebra and it acts on $\mathbf{Cay}^n$ by $n\alpha = \alpha \oplus \cdots \oplus \alpha$, where α is the action on a coordinate.

Let $\{e_1, \ldots, e_7\}$ be an orthonormal basis of $\operatorname{Im}\mathbf{Cay}$. Alternative algebras satisfy the identity $(xa)b + (xb)a = x(ab + ba)$, so

$$r(e_i)\,r(e_j) + r(e_j)\,r(e_i) = r(e_ie_j + e_je_i) = -2\delta_{ij},$$

where δ_{ij} is 0 for $i \neq j$, 1 for $i = j$. Now $\{\alpha(e_1), \ldots, \alpha(e_7)\}$ generates a homomorphic image of the Clifford algebra $\mathrm{Cliff}(\mathbf{R}^7)$. But $\mathrm{Cliff}(\mathbf{R}^7) \cong M_8(\mathbf{R}) \oplus M_8(\mathbf{R})$ direct sum of real matrix algebras, so $\dim_R \alpha(\mathbf{A}) \geqslant 64$. Since α has degree 8 as real representation, now $\alpha(\mathbf{A})$ consists of all **R**-linear transformations of **Cay**.

Expressing $\mathbf{Cay}^n = \mathbf{C}^n \otimes \mathbf{R}^8$ we have $\mathbf{A}$ as the matrix algebra $M_8(\mathbf{R})$ acting on the second factor. Thus $\mathbf{A}$ has $\mathbf{R}$-linear commuting ring $M_n(\mathbf{R})$ acting on the first factor. That proves our assertion in the case $m = n$.

For the general case, define $\tilde{T}: \mathbf{Cay}^{m+n} \to \mathbf{Cay}^{m+n}$ by

$$\tilde{T}(z_1, ..., z_{m+n}) = (0, ..., 0; T(z_1, ..., z_n))$$

and express $\tilde{T}$ as the transformation for a real matrix. Q.E.D.

Lemma 8.5 shows that the unitary group (8.3) is

$$\mathbf{U}(p, q; \mathbf{Cay}) = \{A \otimes I_8 : A \in \mathbf{O}(p, q)\}, \text{ which we denote } \mathbf{O}(p, q) \otimes I_8 . \quad (8.6)$$

Here $\otimes$ means $\otimes_R$ and I_8 is the 8×8 identity matrix. $\mathbf{O}(p, q) \otimes I_8$ acts by automorphisms on $N_{p,q,\mathbf{Cay}}$ by $g(w, z) = (w, g(z))$. Thus we have the semidirect product group

$$G_{p,q,\mathbf{Cay}} = N_{p,q,\mathbf{Cay}} \cdot \mathbf{U}(p, q; \mathbf{Cay}) = N_{p,q,\mathbf{Cay}} \cdot \mathbf{O}(p, q) \otimes I_8 , \quad (8.7a)$$

whose multiplication law is

$$(w_0, z_0, g_0)(w, z, g) = (w_0 + w + \operatorname{Im} h(z_0, g_0(z)), z_0 + g_0(z), g_0 g). \quad (8.7b)$$

We are going to form an extension of $G_{p,q,\mathbf{Cay}}$, more or less as we did over $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{Q}$, but to do this we need some trivialities on Cayley numbers: if $a, b \in \operatorname{Im} \mathbf{Cay}$ and $u, v \in \mathbf{Cay}$ then we need

$$a(b(\bar{a}u) = (ab\bar{a})u \qquad \text{and} \qquad (au)(\bar{a}\bar{v}) = a(u\bar{v})\bar{a}.$$

Both assertions are clear if the three relevant numbers lie in a quaternion subalgebra of $\mathbf{Cay}$, and both are linear in u, v and b and homogeneous in a. Thus we may take $\{a, b, ab, u\}$ orthonormal in $\operatorname{Im} \mathbf{Cay}$ to check the first assertion, and $\{u, v, a\}$ orthonormal in $\operatorname{Im} \mathbf{Cay}$ for the second. Suppose that $\{e_1, ..., e_7\}$ is the orthonormal basis of $\operatorname{Im} \mathbf{Cay}$ in which

$$e_1 e_2 = e_3 , \; e_3 e_4 = e_7 , \; e_2 e_5 = e_7 , \; e_1 e_4 = e_5 ,$$
$$e_2 e_6 = e_4 , \; e_3 e_5 = e_6 , \; e_1 e_6 = e_7 .$$

For the first assertion, we apply an automorphism of $\mathbf{Cay}$ and may assume $a = e_1$, $b = e_2$ and $u = e_4$, and then

$$a(b(\bar{a}u)) = e_1(e_2(-e_1 e_4)) = e_1(-e_2 e_5) = -e_1 e_7 = e_6 = -e_2 e_4 = (ab\bar{a})u.$$

For the second, we apply an automorphism of **Cay** and may assume $u = e_1$, $v = e_2$ and $a = a_3e_3 + a_4e_4$, and then

$$\begin{aligned}(au)(\bar{a}\bar{v}) &= (a_3e_2 - a_4e_5)(a_3e_1 - a_4e_6) = (a_4^2 - a_3^2)\, e_3 - 2a_3a_4e_4 \\ &= (a_3 + a_4e_7)(-a_3e_3 - a_4e_4) = \{a(u\bar{v})\}\bar{a} = a(u\bar{v})\bar{a}.\end{aligned}$$

We reformulate the identities just proved, in group-theoretic terms:

LEMMA 8.8. *Let* **Spin**(7) *denote the* (2-*sheeted*) *simply connected covering group of the rotation group* **SO**(7). *Then we can realize*

σ: **Spin**(7) $\rightarrow$ **SO**(8), *spin representation, by action on* **Cay**

and the covering as

ν: **Spin**(7) $\rightarrow$ **SO**(7), *vector representation, by action on* Im**Cay**

in such a way that

$$\mathrm{Im}\{\sigma(a)\, u \cdot \overline{\sigma(a)v}\} = \nu(a) \cdot \mathrm{Im}(u\bar{v}) \qquad \textit{for} \quad a \in \mathbf{Spin}(7) \qquad \textit{and} \qquad u, v \in \mathbf{Cay}.$$

Proof. If $a \in$ **Cay** let $l(a)$: **Cay** $\rightarrow$ **Cay** by $z \rightarrow az$. As in the proof of Lemma 8.5, l(Im **Cay**) generates an associative algebra $\mathbf{A} \cong M_8(\mathbf{R})$, homomorphic image of the Clifford algebra

$$\mathrm{Cliff}(\mathbf{R}^7) \cong M_8(\mathbf{R}) \oplus M_8(\mathbf{R}),$$

and that homomorphism p: $\mathrm{Cliff}(\mathbf{R}^7) \rightarrow \mathbf{A}$ maps the generating subspace $\mathbf{R}^7$ onto l(Im **Cay**). One constructs **Spin**(7) as the multiplicative group of all invertible $x \in \mathrm{Cliff}(\mathbf{R}^7)$ such that

$$|\, x \,| = 1, \; x \cdot \mathbf{R}^7 \cdot x^{-1} = \mathbf{R}^7,$$

and

$$\nu(x)\colon \mathbf{R}^7 \rightarrow \mathbf{R}^7 \text{ by } y \rightarrow xyx^{-1} \text{ has } \det \nu(x) = 1.$$

Then ν is the universal covering **Spin**(7) $\rightarrow$ **SO**(7), the vector representation. Here $p\,|_{\mathbf{Spin}(7)}$ is an irreducible representation σ, the spin representation, which thus is given as an **R**-linear action on **Cay**.

The first identity proved above, says $l(a) \cdot l(b) \cdot l(\bar{a}) = l(ab\bar{a})$ for $a, b \in$ Im **Cay**. Let U be the multiplicative group generated by

$$\{l(a)\colon a \in \mathrm{Im}\,\mathbf{Cay},\; |\, a \,| = 1\}.$$

Now $a \cdot l(\mathrm{Im}\,\mathbf{Cay}) \cdot a^{-1} = l(\mathrm{Im}\,\mathbf{Cay})$ for all $a \in U$, it follows that U is a nontrivial closed connected subgroup of the simple group

$\sigma(\mathbf{Spin}(7))$, and we conclude that $\sigma\colon \mathbf{Spin}(7) \cong U$. We identify $\mathbf{Spin}(7)$ to U under this isomorphism, and now the vector representation is given on generators of $\mathbf{Spin}(7)$ by $\nu(l(a))\colon l(b) \to l(ab\bar{a})$ for $a, b \in \operatorname{Im} \mathbf{Cay}$, $|a| = 1$. Re-write that as a representation of $\mathbf{Spin}(7)$ on $\operatorname{Im} \mathbf{Cay}$ given on generators by $\nu(l(a))\colon b \to ab\bar{a}$ for $a, b \in \operatorname{Im} \mathbf{Cay}$ with $|a| = 1$.

Taking imaginary component in the second identity proved above, we have $\operatorname{Im}\{l(a)\, u \cdot \overline{l(a)v}\} = \nu(l(a)) \cdot \operatorname{Im}(u\bar{v})$ as $l(a)$ runs over our generating set for $\mathbf{Spin}(7)$. Now $\operatorname{Im}\{\sigma(a)\, u \cdot \overline{\sigma(a)v}\} = \nu(a) \cdot \operatorname{Im}(u\bar{v})$ for all $a \in \mathbf{Spin}(7)$. Q.E.D.

Lemma 8.8 tells us that $\mathbf{Spin}(7)$ acts by automorphisms on $N_{p,q,\mathbf{Cay}}$, by the formula

$$a(w; z_1, \ldots, z_{p+q})\, a^{-1} = (\nu(a)\, w; \sigma(a)\, z_1, \ldots, \sigma(a)\, z_{p+q}). \tag{8.9a}$$

The action (8.9a) of $\mathbf{Spin}(7)$ commutes with the action of

$$\mathbf{U}(p, q; \mathbf{Cay}) = \mathbf{O}(p, q) \otimes I_8 ;$$

in fact it takes place on the other factor of $\mathbf{Cay}^{p,q} = \mathbf{R}^{p,q} \otimes \mathbf{R}^8$. Thus we have the larger group

$$G_{p,q,\mathbf{Cay}} \cdot \mathbf{Spin}(7) = N_{p,q,\mathbf{Cay}} \cdot \{\mathbf{O}(p, q) \times \mathbf{Spin}(7)\}. \tag{8.9b}$$

It has group law

$$\begin{aligned}(w_0, z_0, g_0, a_0)&(w, z, g, a)\\ &= (w_0 + a_0 w \bar{a}_0 + \operatorname{Im} h(z_0, g_0 a_0 z_0), z_0 + g_0 a_0 z, g_0 g, a_0 a).\end{aligned} \tag{8.9c}$$

If instead we let a range over $\mathbf{Spin}(7) \times \mathbf{R}^+$, then we get parabolic-type group

$$\begin{aligned}P_{p,q,\mathbf{Cay}} &= G_{p,q,\mathbf{Cay}} \cdot \{\mathbf{Spin}(7) \times \mathbf{R}^+\}\\ &= N_{p,q,\mathbf{Cay}} \cdot \{\mathbf{O}(p, q) \times \mathbf{Spin}(7) \times \mathbf{R}^+\},\end{aligned} \tag{8.10a}$$

in which the analog of the Langlands decomposition is $P = MAN$ with

$$N = N_{p,q,\mathbf{Cay}}, \qquad M = \mathbf{O}(p, q) \times \mathbf{Spin}(7), \qquad \text{and} \qquad A = \mathbf{R}^+. \tag{8.10b}$$

As remarked (or hinted) earlier, $P_{1,0,\mathbf{Cay}}$ is the minimal parabolic subgroup in the simply connected group of type F_4 and real rank 1.

The classes $[\eta] \in \hat{N}_{p,q,\mathbf{Cay}}$ which annihilate the center $\operatorname{Im} \mathbf{Cay}$ are just the unitary characters

$$\chi_v(w, z) = e^{i \operatorname{Re} h(v,z)}, \tag{8.11a}$$

where $v \in \mathbf{Cay}^{p,q}$. As before, χ_v has stabilizer

$$L_v{}^S = \{(g, a) \in \mathbf{O}(p, q) \times S: g(v) = \sigma(a)v\} \tag{8.11b}$$

in $\mathbf{O}(p, q) \times S$ for any subgroup S of $\mathbf{Spin}(7) \times \mathbf{R}^+$, and χ_v extends to $N_{p,q,\mathbf{Cay}} \cdot \{\mathbf{O}(p, q) \times S\}$ by the formula

$$\tilde{\chi}_v(w, z, g, a) = \chi_v(w, z) = e^{i \operatorname{Re} h(v,z)}. \tag{8.11c}$$

This gives us representation classes

$$[\pi_{v,\gamma,\beta}] = [\operatorname{Ind}_{N_{p,q,\mathbf{Cay}} L_v{}^S \uparrow G_{p,q,\mathbf{Cay}} \cdot S}(\tilde{\chi}_v \otimes \gamma \otimes \beta)] \in (G_{p,q,\mathbf{Cay}} \cdot S)^\wedge, \tag{8.12}$$

where $[\gamma] \in \{g \in \mathbf{O}(p, q): g(v) \in \sigma(S)v\}^\wedge$, and $[\beta] \in \hat{S}$. The classes $[\pi_{v,\gamma,\beta}]$ with v zero, positive, negative, or isotropic correspond to the classes in the series (1), (2), (3), and (4) of Proposition 7.8. To be more specific one must (i) enumerate the $\mathbf{O}(p, q) \times S$-orbits on $\mathbf{Cay}^{p,q}$ and then (ii) calculate $L_v{}^S$ for a choice of v in each of those orbits. If $p + q = 1$ this is just the matter of the $\sigma(S)$-orbit structure of $\mathbf{Cay} \cong \mathbf{R}^8$, but if $p + q > 1$ it is messy.

The classes $[\eta] \in \hat{N}_{p,q,\mathbf{Cay}}$ which do not annihilate the center $\operatorname{Im} \mathbf{Cay}$, are in bijective correspondence $[\eta_\lambda] \leftrightarrow \lambda$ with the nonzero $\mathbf{R}$-linear $\lambda: \operatorname{Im} \mathbf{Cay} \to \mathbf{R}$, as in Lemma 4.4, by $\eta_\lambda(w, z) = e^{i\lambda(w)}\eta_\lambda(0, z)$. The $\mathbf{O}(p, q) \times S$-stabilizer of $[\eta_\lambda]$ is

$$L^S_{[\eta_\lambda]} = \mathbf{O}(p, q) \times \{a \in S: \lambda(w) = \lambda(aw\bar{a}) \quad \text{for} \quad w \in \operatorname{Im} \mathbf{Cay}\}. \tag{8.13a}$$

If we represent

$$\lambda: \operatorname{Im} \mathbf{Cay} \to \mathbf{R} \text{ by } \lambda(w) = -\operatorname{Re}(l\bar{w}), \qquad 0 \neq l \in \operatorname{Im} \mathbf{Cay}, \tag{8.13b}$$

then as in Section 7 we have

$$L^S_{[\eta_\lambda]} = \mathbf{O}(p, q) \times S_\lambda, \tag{8.13c}$$

where $S_\lambda = \{a \in S \cap \mathbf{Spin}(7): \nu(a)l = l\}$. We check that $[\eta_\lambda]$ extends to $N_{p,q,\mathbf{Cay}} \cdot L^S_{[\eta_\lambda]}$:

LEMMA 8.14. *$[\eta_\lambda]$ extends to $N_{p,q,\mathbf{Cay}} \cdot \{\mathbf{O}(p, q) \times \mathbf{Spin}(7)_\lambda\}$.*

Proof. Let $Z_\lambda = \{w \in \operatorname{Im} \mathbf{Cay}: \lambda(w) = 0\}$, so $[\eta_\lambda]$ factors through a representation class $[\bar{\eta}_\lambda]$ of $N_{p,q,\mathbf{Cay}}/Z_\lambda \cong N_{4p,4q,C}$. As in (4.15), realize $[\bar{\eta}_\lambda]$ by the representation $\bar{\eta}_\lambda^{0,s}$ of $N_{4p,4q,C}$ on a square integrable cohomology group $\mathbf{H}_2^{0,s}(\mathscr{L}_\lambda)$, so $[\eta_\lambda]$ is realized by the lift $\eta_\lambda^{0,s}$ of $\bar{\eta}_\lambda^{0,s}$ to $N_{p,q,\mathbf{Cay}}$.

View $\mathbf{Cay}^{p,q}$ as a complex vector space where the scalars are left multiplications by elements of $\mathbf{R} + \mathbf{R}l$. Then the projection

$$N_{p,q,\mathbf{Cay}} \to N_{p,q,\mathbf{Cay}}/Z_\lambda = N_{4p,4q,C}$$

induces a $\mathbf{C}$-linear isomorphism of $\mathbf{Cay}^{p,q}$ onto $\mathbf{C}^{4p,4q}$. This carries $\mathbf{O}(p, q) \otimes I_8$ to a subgroup of $\mathbf{O}(4p, 4q) \otimes I_2 \subset \mathbf{U}(4p, 4q)$. It carries $\mathbf{Spin}(7)_\lambda = \mathbf{Spin}(6)$ to a subgroup $I_{p+q} \otimes \mathbf{SU}(4)$ of $\mathbf{U}(4p, 4q)$. Now the ingredients of $\eta_\lambda^{0,s}$ all are invariant under $\mathbf{O}(p, q) \times \mathbf{Spin}(7)_\lambda$, and this gives the extension. Q.E.D.

The extensions $[\tilde{\eta}_\lambda]$ of Lemma 8.14 give us representation classes

$$[\pi_{\lambda,\gamma,\sigma}] = [\mathrm{Ind}_{N_{p,q,\mathbf{Cay}}\cdot\{\mathbf{O}(p,q)\times S_\lambda\}\uparrow G_{p,q,\mathbf{Cay}}\cdot S}(\tilde{\eta}_\lambda \otimes \gamma \otimes \sigma)] \in (G_{p,q,\mathbf{Cay}} \cdot S)^\wedge, \tag{8.15}$$

where $[\gamma] \in \mathbf{O}(p, q)^\wedge$ and $\sigma \in \hat{S}_\lambda$. These classes $[\pi_{\lambda,\gamma,\sigma}]$ correspond to the classes in series (5) of Proposition 7.8. They are parameterized by $(\mathrm{Im}\,\mathbf{Cay} - \{0\})/\nu(S) \times \mathbf{O}(p, q)^\wedge \times \hat{S}_\lambda$ under $[\pi_{\lambda,\gamma,\sigma}] \leftrightarrow (\nu(S)^*\lambda, [\gamma], \sigma)$.

Every class in $(G_{p,q,\mathbf{Cay}} \cdot S)^\wedge$ is given by (8.12) or (8.15).

Plancherel measure is concentrated on the classes (8.15). Thus, for example, $G_{p,q,\mathbf{Cay}}$ has Plancherel formula

$$\int_{G_{p,q,\mathbf{Cay}}} |f(x)|^2\, dx = c \int_{\mathbf{R}^7} \left\{ \int_{[\gamma]\in\mathbf{O}(p,q)^\wedge} \|(\tilde{\eta}_\lambda \otimes \gamma)(f)\|_2^2\, d[\gamma] \right\} \| l \|^{4(p+q)}\, dl, \tag{8.16}$$

where $\lambda(w) = -\mathrm{Re}(l\bar{w})$ and $l \in \mathrm{Im}\,\mathbf{Cay}$ is viewed as an element of $\mathbf{R}^7$. Similarly $N_{p,q,\mathbf{Cay}} \cdot \{\mathbf{O}(p, q) \times \mathbf{Spin}(7)\} = G_{p,q,\mathbf{Cay}} \cdot \mathbf{Spin}(7)$ has Plancherel formula

$$\left\{ \begin{aligned} &\int_{G_{p,q,\mathbf{Cay}}\cdot\mathbf{Spin}(7)} |f(x)|^2\, dx \\ &\quad = c \int_0^\infty \left\{ \int_{[\gamma]\in\mathbf{O}(p,q)^\wedge} \sum_{[\sigma]} \| \pi_{\lambda_r,\gamma,\sigma}(f)\|_2^2 \deg(\sigma)\, d[\gamma] \right\} r^{4(p+q)}\, dr, \end{aligned} \right. \tag{8.17}$$

where λ_r: $\mathrm{Im}\,\mathbf{Cay} \to \mathbf{R}$ by $\lambda_r(w) = -\mathrm{Re}(ri\bar{w})$, $r > 0$, so $\mathbf{Spin}(7)_{\lambda_r} = \mathbf{Spin}(6) \cong \mathbf{SU}(4)$, and $[\sigma]$ runs over $\hat{S}_{\lambda_r} \cong \mathbf{SU}(4)^\wedge$.

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